MAGIC: Ergodic Theory Lecture 5 - Recurrence & Ergodicity

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February 20th 2013

In this lecture we discuss recurrence properties of measure preserving transformations.
We will state Birkhoff's Ergodic Theorem and study some simple applications.

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Proof (sketch):

Let

$$A_n = \bigcup_{k \geq n} T^{-k} A = \{x \in X \mid \text{the orbit of } x \text{ lies in } A \text{ at some time} \geq n\} .$$

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Before we discuss ergodic theorems we need the notion of conditional expectation.

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such that

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Idea: $\mathbb{E}(f|\mathcal{A})$ - the conditional expectation of f given $\mathcal{A}$ - is the unique $\mathcal{A}$-measurable function that best approximates f.

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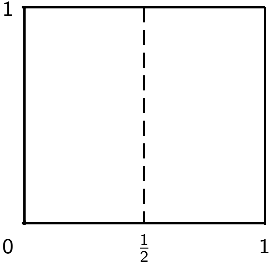

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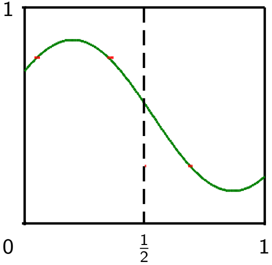

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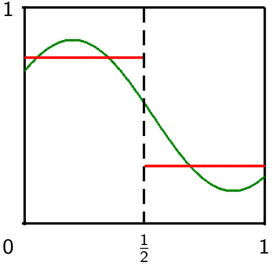

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Note: if T is ergodic then $\mathcal{I} = \{\emptyset, X\}$ a.e. and $E(f \mid \mathcal{I}) = \int f \, d\mu$.

Von Neumann's Ergodic Theorem

Let $(X, \mathcal{B}, \mu)$ be a probability space and let $T : X \to X$ be an mpt. Let $\mathcal{I}$ denote the σ-algebra of invariant sets. Let $f \in L^2(X, \mathcal{B}, \mu)$.

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Corollary

If, in addition, T is ergodic, then

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Sketch of a proof for von Neumann's Ergodic Theorem

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The proof is hard analysis!

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What is the expected first return time?

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Remark

The expected return time of a point $x \in A$ to A is

$$\frac{1}{\mu(A)} \int_A n_A \, d\mu = \frac{1}{\mu(A)}.$$

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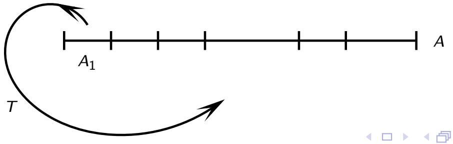

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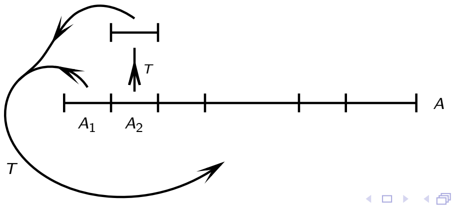

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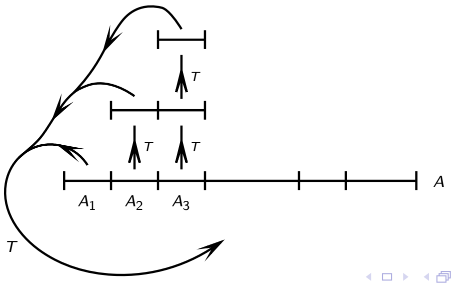

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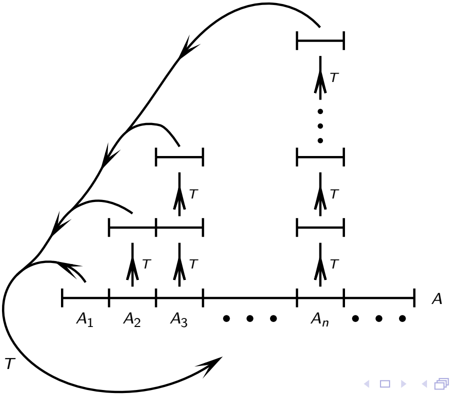

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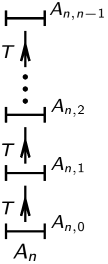

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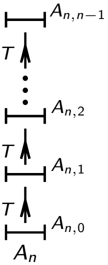

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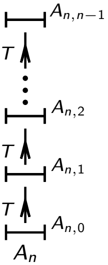

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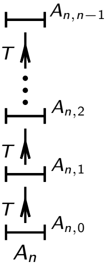

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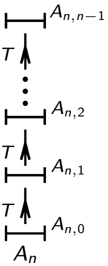

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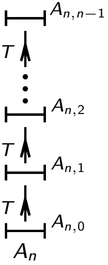

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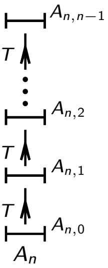

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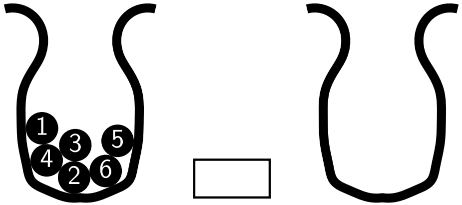

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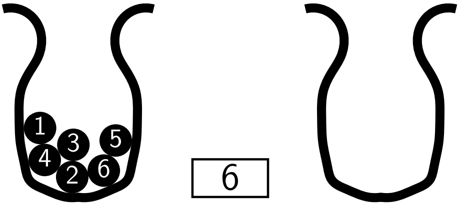

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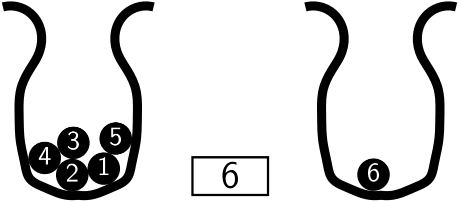

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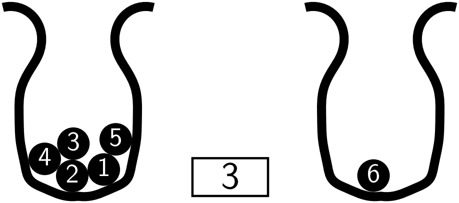

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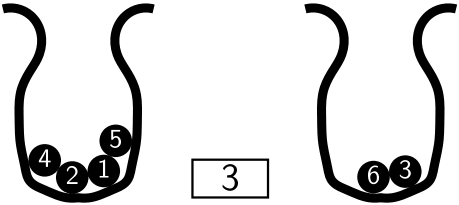

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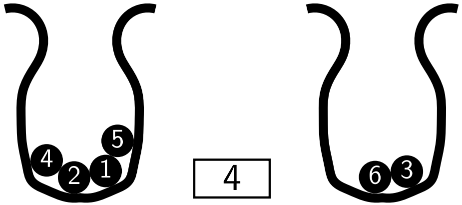

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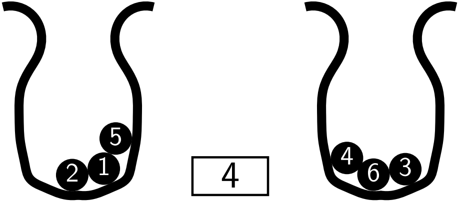

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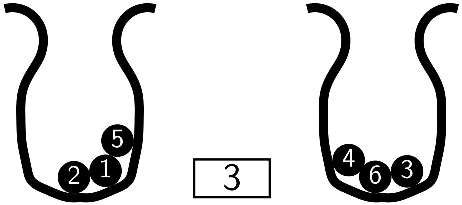

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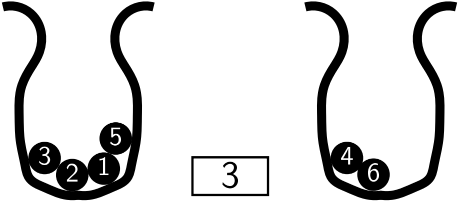

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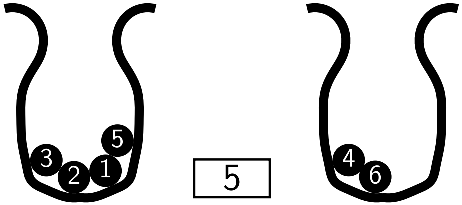

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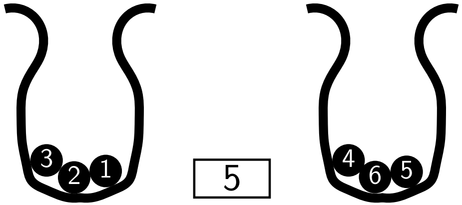

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We model the system as a shift of finite type on 101 symbols with an appropriate Markov measure.
Regard $x_j \in \{0, 1, \ldots, 100\}$ as the number of balls in the first urn after j seconds.

As the number of balls in the first urn either increases or decreases by 1, $(x_j)_{j=0}^{\infty} \in \Sigma_A^+$ where

$$A = \begin{pmatrix} 0 & 1 & 0 & \multicolumn{2}{c}{\dots\dots\dots} & 0 \\ 1 & 0 & 1 & \ddots & & \vdots \\ 0 & 1 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & 1 & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0 & 1 \\ 0 & \multicolumn{2}{c}{\dots\dots\dots} & 0 & 1 & 0 \end{pmatrix}.$$

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Let $P = (P_{ij})$ a stochastic matrix.
One can easily check $pP = p$.
Hence we get a Markov measure μ_P.
Let $A = [100]$ - a cylinder of length 1. This represents there being 100 balls in the first urn. By Kac's Lemma, the expected first return time to A is

$$\begin{aligned} \frac{1}{\mu_P(A)} &= 2^{100} \text{ seconds} \\ &= 4 \times 10^{22} \text{ years} \\ &= \text{about } 3 \times 10^{12} \times \text{ age of the Universe!} \end{aligned}$$

Open problem: how are return times distributed? We would expect a Poisson distribution. This is known for a number of examples (hyperbolic dynamics,...)

Normality of numbers

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Definition

$x \in [0, 1]$ is *(simply) normal in base r* if it has a unique base r expansion and for each digit $k \in \{0, 1, \ldots, r-1\}$, the frequency with which the digit k occurs in the base r expansion of x is $\frac{1}{r}$.

Proposition

Lebesgue-a.e. $x \in [0, 1]$ is simultaneously normal in every base $r \geq 2$.

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$$x = \sum_{j=1}^{\infty} \frac{x_j}{r^j}$$

then

$$x_j = k \Leftrightarrow T^{j-1} x \in \left[\frac{k}{r}, \frac{k+1}{r}\right).$$

Hence

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It is easy to give an example of an $x \in [0,1]$ that is (simply) normal in a given base r.

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Remark

It is easy to give an example of an $x \in [0,1]$ that is (simply) normal in a given base r. However, there is no known example of an x that is simultaneously normal in every base.

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3. Gauss' measure μ and Lebesgue measure λ are equivalent: they have the same sets of measure zero.

Proposition

For Lebesgue a.e. $x \in [0, 1]$, the frequency with which digit $k \in \mathbb{N}$ occurs in the continued fraction expansion of x is

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As Gauss' measure and Lebesgue measure are equivalent, this holds Lebesgue a.e.

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2. the geometric mean of the continued fraction digits of x is:
$$\sqrt[n]{x_0 \ldots x_{n-1}} \longrightarrow \prod_{k=1}^{\infty} \left(1 + \frac{1}{k(k+2)}\right)^{\frac{\log k}{\log 2}} < \infty \text{ a.e.}$$

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Next lecture

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In the next lecture, we study this question in the context of continuous transformations of compact metric spaces.