MAGIC: Ergodic Theory Lecture 4 - Ergodicity and Mixing

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February 13th 2013

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Then

$$\frac{1}{n}\sum_{j=0}^{n-1} f(T^j x) \longrightarrow \int f \, d\mu \ \ \mu\text{-a.e. } x \in X.$$

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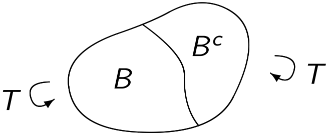

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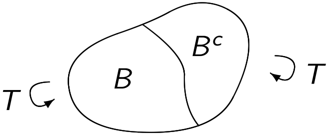

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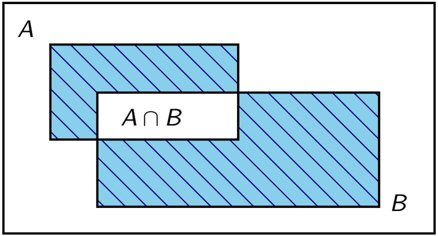

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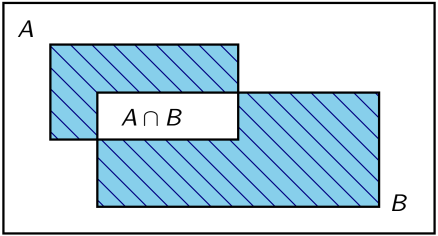

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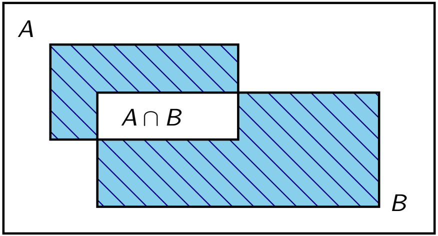

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Remark: We can replace L^1 in 2. by L^2.

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$\implies$ Suppose $\alpha = \frac{p}{q}$, $p, q \in \mathbb{Z}$ with $q \neq 0$. Let $f(x) = \exp(2\pi iqx)$. Then f is non-constant and

$$\begin{aligned} f(Tx) &= \exp\left(2\pi iq\left(x + \frac{p}{q}\right)\right) = \exp(2\pi i(qx + p)) \\ &= \exp(2\pi iqx) = f(x) \end{aligned}$$

so f is T-invariant. Hence T is not ergodic.

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If $n \neq 0$, then $\exp(2\pi i n\alpha) \neq 1$ (as α irrational). Hence: $n \neq 0 \implies c_n = 0$. Hence f has Fourier series c_0, i.e. f is constant a.e.

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In higher dimensions this is: If $f \in L^2$ has Fourier series

$$\sum_{n \in \mathbb{Z}^k} c_n e^{2\pi i \langle n,x \rangle}$$

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Suppose $n \neq 0$. Then $2^p n \to \infty$ as $p \to \infty$. By the Riemann-Lebesgue lemma: $c_n = c_{2^p n} \to 0$. Hence $c_n = 0 \, \forall n \neq 0$.
Hence f has Fourier series c_0 so f is constant a.e. Hence T is ergodic.

Toral endomorphisms

Let $X = \mathbb{R}^k/\mathbb{Z}^k$ and A be a $k \times k$ integer matrix s.t. $\det A \neq 0$. Define $T : X \to X$ by $Tx = Ax \mod 1$. We know that T preserves Lebesgue measure.

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Let T be a measure-preserving transformation of a probability space $(X, \mathcal{B}, \mu)$. Suppose $\mathcal{A}$ is an algebra that generates $\mathcal{B}$. We show how to prove ergodicity by approximating invariant sets $T^{-1}B = B$ by sets in $\mathcal{A}$.

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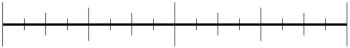

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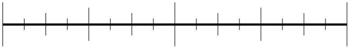

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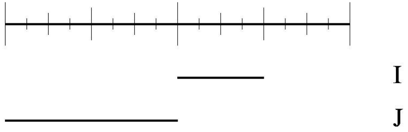

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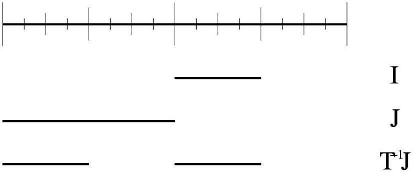

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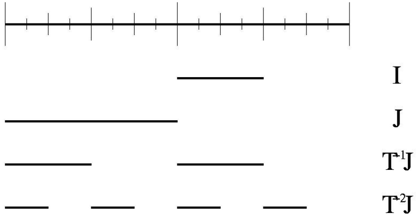

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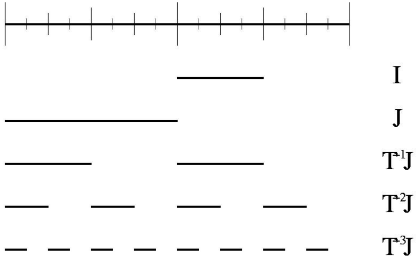

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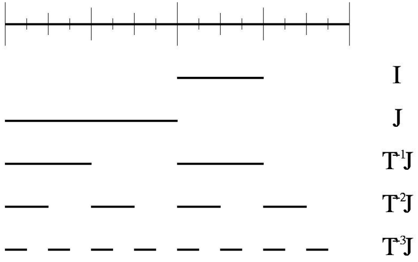

$\mu(I \cap J) = 0$, $\mu(I \cap T^{-1}J) = 1/4$, $\mu(I \cap T^{-2}J) = 1/8$, $\mu(I \cap T^{-3}J) = 1/8$.

If n is large enough then $\mu(I \cap T^{-n}J) = \mu(I)\mu(J)$ for all dyadic intervals I, J.

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Hence $\mu(B) = 0$ or 1 by the technical lemma.

$\square$

Bernoulli Shifts

Let $\Sigma_k = \left\{ (x_j)_{j=0}^{\infty} \mid x_j \in \{1, \ldots, k\} \right\}$ be the full one-sided k-shift.
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Let $p = (p_1, \ldots, p_k)$ be a probability vector. Recall the p-Bernoulli measure μ_p defined on cylinders by

$$\mu_p[i_0, \ldots, i_{n-1}] = p_{i_0} \ldots p_{i_{n-1}}.$$

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The same proof as for the doubling map then works.

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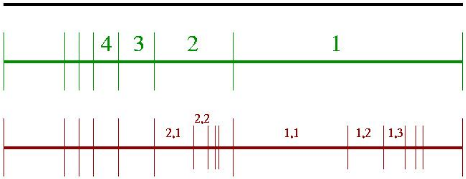

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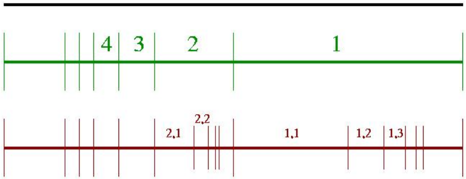

Call $I(x_0, \ldots, x_n)$ the cylinder of rank n that contains x.

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By the technical lemma, T is ergodic wrt μ.

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Recall Birkhoff's Ergodic Theorem: Let T be an ergodic mpt of $(X, \mathcal{B}, \mu)$ and let $f \in L^1(X, \mathcal{B}, \mu)$. Then

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$$T \text{ is ergodic} \Leftrightarrow \forall A, B \in \mathcal{B}, \ \frac{1}{n} \sum_{j=0}^{n-1} \mu(T^{-j} A \cap B) \to \mu(A)\mu(B).$$

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There are examples to show that neither of these inequalities can be reversed.

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Thus T is strong-mixing if and only if the events $T^{-n}A$ and B become independent as $n \to \infty$.
T is weak-mixing (or ergodic) if the events $T^{-n}A, B$ become independent as $n \to \infty$ in the absolute Cesàro (or Cesàro) sense.

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$$\langle Uf, Ug \rangle = \int f \circ T \overline{g \circ T} \, d\mu = \int (f \bar{g}) \circ T \, d\mu = \int f \bar{g} \, d\mu = \langle f, g \rangle.$$

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Theorem

T is weak-mixing $\Leftrightarrow$ 1 is the only eigenvalue for U.

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Indeed, let $f(x) = e^{2\pi ix}$. Then

$$f(Tx) = e^{2\pi(x+\alpha)} = e^{2\pi i\alpha} f(x)$$

so that $e^{2\pi i\alpha}$ is an eigenvalue.

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Example

The doubling map equipped with Lebesgue measrure is Bernoulli. It is isomorphic to the Bernoulli $(1/2, 1/2)$-shift via the coding map $\pi : \Sigma_2 \to [0, 1]$, $\pi((x_j)) = \sum_{j=0}^{\infty} \frac{x_j}{2^{j+1}}$.

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(And none of these implications can be reversed.)
There are many other forms of mixing (mild-mixing, r-fold mixing) that can be fitted in to this scheme.

Next lecture

In the next lecture we look at Birkhoff's Ergodic Theorem and recurrence.