MAGIC: Ergodic Theory Lecture 3 - Invariant Measures

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February 6th 2013

In this lecture:

- we give some basics about measure theory
- define and study invariant measures and measure-preserving transformations.

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3. Let X be a compact metric space. The Borel σ-algebra is the smallest σ-algebra that contains every open set.

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(Example: a.e. real number is irrational w.r.t. Lebesgue measure - the rationals have measure zero.)

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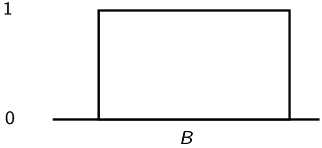

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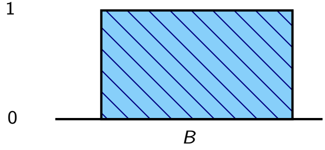

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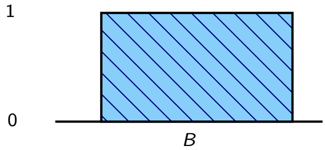

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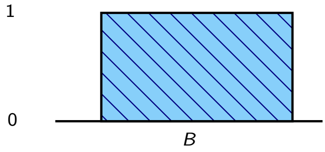

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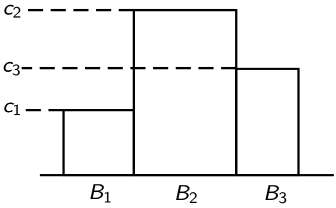

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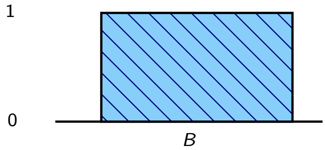

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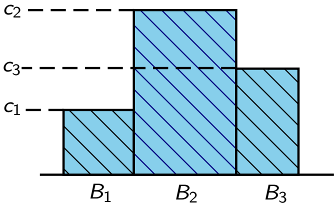

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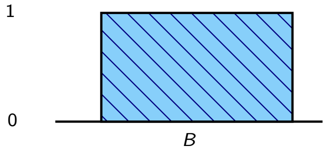

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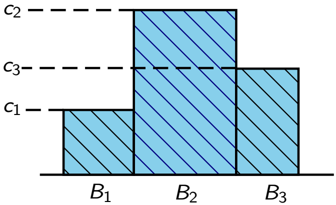

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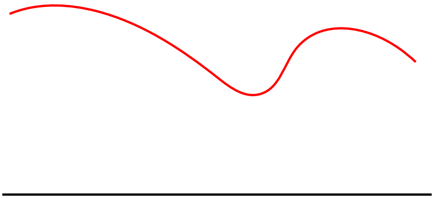

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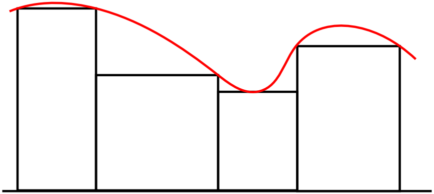

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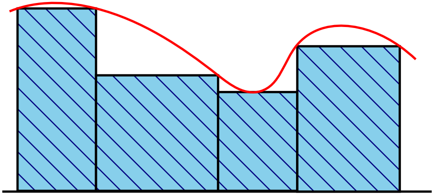

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We can also work with complex functions by taking real and imaginary parts.

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Then: there exists a *unique* measure $\mu : \mathcal{B}(\mathcal{A}) \longrightarrow \mathbb{R}^+ \cup \{\infty\}$ that extends $\mu : \mathcal{A} \longrightarrow \mathbb{R}^+ \cup \{\infty\}$.

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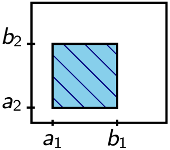

The Kolmogorov Extension Theorem gives k-dimensional Lebesgue measure on the k-dimensional torus.

$$\mu(\text{rectangle}) = (b_1 - a_1) \times (b_2 - a_2)$$

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$\rho(x) = x$	gives Lebesgue measure.
$\rho(x) = \frac{1}{\log 2} \int_0^x \frac{dt}{1+t}$	gives Gauss' measure (will use this later when studying continued fractions)

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$\mu_\rho[a,b] = \rho(b) - \rho(a)$

The Kolmogorov Extension theorem gives a measure on the Borel σ-algebra.

Examples

$\rho(x) = x$	gives Lebesgue measure.
$\rho(x) = \frac{1}{\log 2}\int_0^x \frac{dt}{1+t}$	gives Gauss' measure (will use this later when studying continued fractions)

We illustrate that Stieltjes measures can be surprisingly complicated.

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Fact: If $\rho'(x)$ is continuous then $\mu_\rho \ll$ Lebesgue. If in addition, $\rho'(x) > 0$, then μ_ρ and Lebesgue measure are equivalent. Gauss' measure and Lebesgue measure are equivalent ($\rho'(x) = \frac{1}{\log 2} \frac{1}{1+x}$).

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Invariant Measures

Let $(X, \mathcal{B}, \mu)$ be a probability space.

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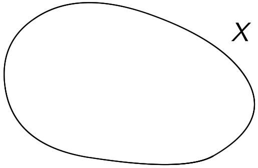

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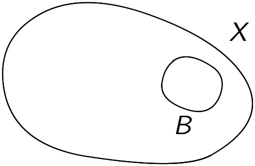

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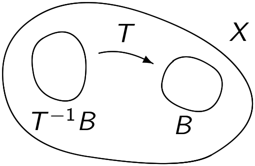

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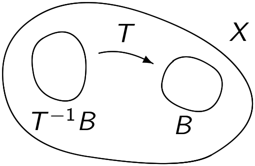

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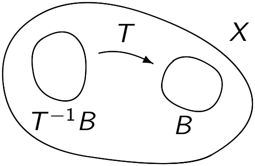

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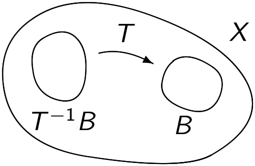

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Remark: We can replace L^1 by L^2.

Proof (sketch):

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- Using periodic points.
- Using the Kolmogorov Extension Theorem.
- (When X is a group) using Haar measure.

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$$\begin{aligned}
\int f \circ T \, d\mu &= \frac{1}{n}(f(Tx) + \cdots + f(T^{n-1}x) + f(T^n x)) \\
&= \frac{1}{n}(f(x) + f(Tx) + \cdots + f(T^{n-1}x)) \\
&= \int f \, d\mu.
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Then μ is T-invariant $\iff T_*\mu(B) = \mu(B)\,\forall B \in \mathcal{B}$.
To show $T_*\mu(B) = \mu(B)\,\forall B \in \mathcal{B}$, it is sufficient to prove that $T_*\mu(B) = \mu(B)\,\forall B \in \mathcal{A}$, where $\mathcal{A}$ is an algebra that generates $\mathcal{B}$.
(This is because the Kolmogorov Extension Theorem tells us that a "measure" on $\mathcal{A}$ extends *uniquely* to a measure on $\mathcal{B}$.)

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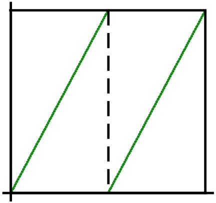

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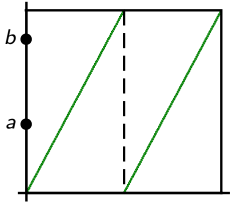

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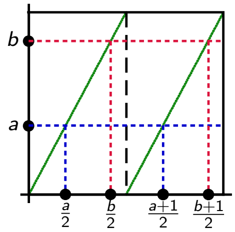

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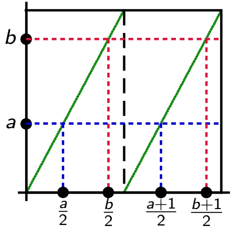

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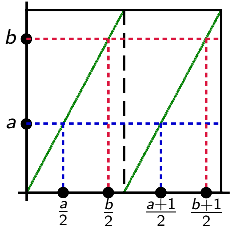

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$$\mu T^{-1}[a,b] = \frac{b}{2} - \frac{a}{2} + \frac{b+1}{2} - \frac{a+1}{2} = b - a = \mu[a,b] \quad \square$$

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Proof:

Again, it is sufficient to prove that $\mu T^{-1}[a,b] = \mu[a,b]$ for every interval $[a,b]$.

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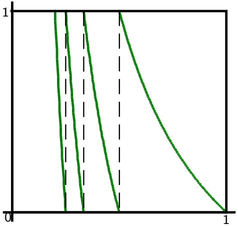

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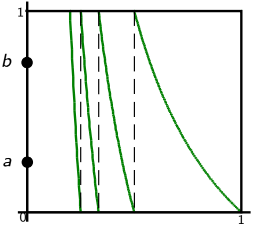

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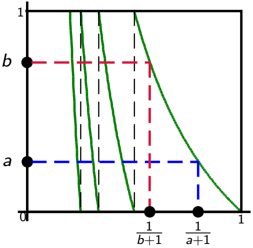

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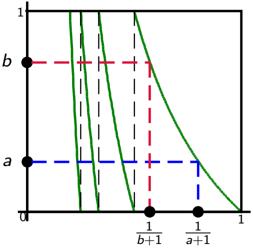

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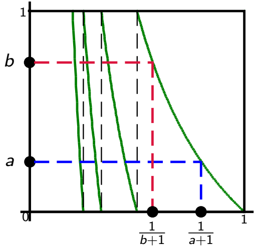

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&= \frac{1}{\log 2}\left(\log\left(1+\frac{1}{a+n}\right) - \log\left(1+\frac{1}{b+n}\right)\right)
\end{aligned}$$

Hence (easy check!)

$$\mu(T^{-1}[a,b]) = \sum_{n=1}^{\infty} \mu\left(\left[\frac{1}{b+n}, \frac{1}{a+n}\right]\right) = \mu([a,b]).$$

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$$[i_0, i_1, \ldots, i_{n-1}, i_n] = \{x = (i_0, i_1, \ldots, i_{n-1}, i_n, *, *, *, \cdots)\}$$

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If $P_{i,j} = P_j$ then

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We call μ a *Bernoulli measure*.

Proposition

Let $\sigma : \Sigma_k \to \Sigma_k$, $(\sigma x)_j = x_{j+1}$, be the full one-sided k-shift, P be a stochastic matrix, and let p be a left probability eigenvector. Then the Markov measure $\mu = \mu_P$ is σ-invariant.

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Proposition

Define $T : \mathbb{R}^k/\mathbb{Z}^k$ by $Tx = x + a \bmod 1$. Then Lebesgue measure is T-invariant.

Group automorphisms

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Hence $T_*\mu$ is invariant under any group rotation. By uniqueness of Haar measure, $T_*\mu$ is Haar measure, i.e. $T_*\mu = \mu$. $\square$

Next lecture

In the next lecture we define ergodic measures.
We will give examples of ergodic measure-preserving transformations.
We will also see how mixing properties of the dynamics imply ergodicity.