MAGIC: Ergodic Theory Lecture 2 - Uniform distribution mod 1

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January 31st 2013

Introduction

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In this lecture we consider the distribution of fractional parts of a sequence of reals.
We also give some applications to number theory.

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$$x_n = [x_n] + \{x_n\}$$

integer part
(largest integer $\leq x_n$)

fractional part
$= x_n \mod 1 \in [0, 1)$

Uniform distribution in dimension 1

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$$x_n = \underbrace{[x_n]}_{\substack{\text{integer part} \\ (\text{largest integer} \leq x_n)}} + \underbrace{\{x_n\}}_{\substack{\text{fractional part} \\ = x_n \mod 1 \in [0,1)}}$$

How are the fractional parts of x_n distributed in $[0, 1)$?

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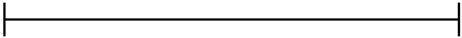

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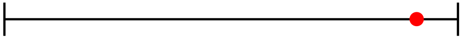

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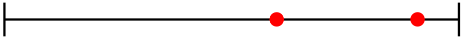

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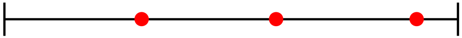

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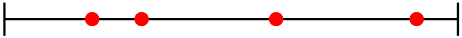

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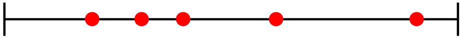

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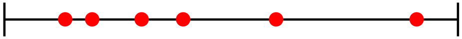

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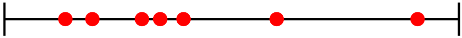

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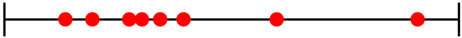

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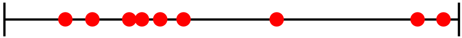

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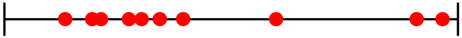

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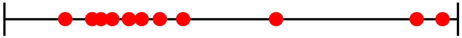

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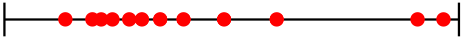

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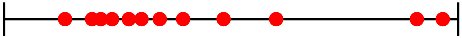

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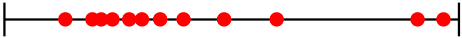

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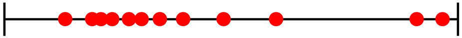

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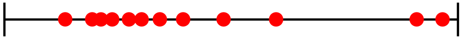

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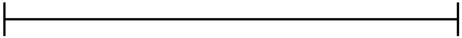

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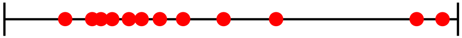

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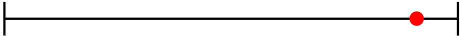

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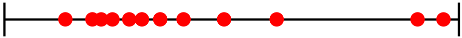

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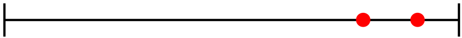

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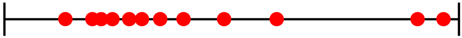

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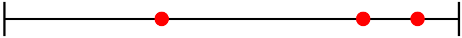

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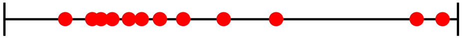

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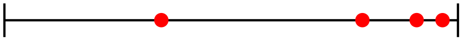

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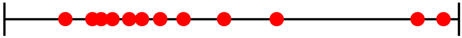

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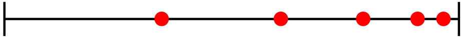

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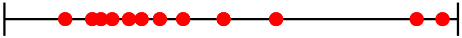

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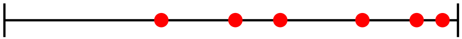

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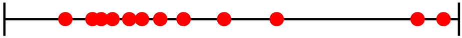

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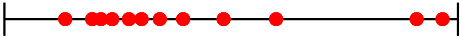

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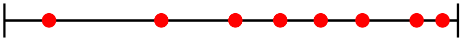

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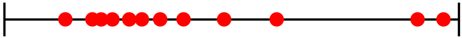

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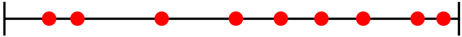

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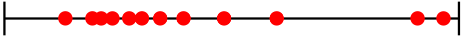

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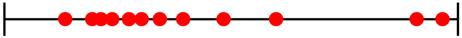

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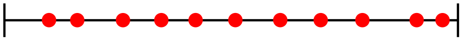

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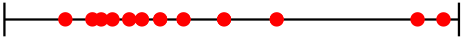

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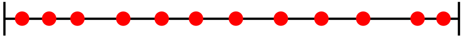

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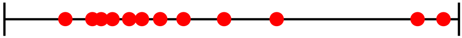

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We need a usable criterion to check whether a sequence is udm1.

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3. $\forall \ell \in \mathbb{Z}\backslash\{0\}$
$$\frac{1}{n}\sum_{j=0}^{n-1} \exp 2\pi i \ell x_j \longrightarrow 0 \text{ as } n \to \infty.$$

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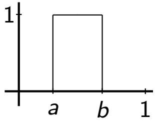

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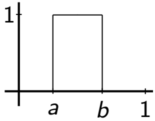

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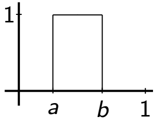

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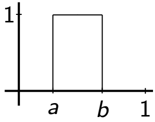

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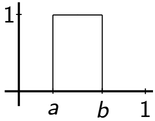

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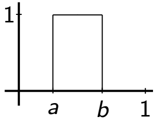

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$$x_n \text{ udm1} \iff \frac{1}{n}\sum_{j=0}^{n-1} \chi_{[a,b]}(\{x_j\}) \longrightarrow b-a = \int \chi_{[a,b]}(x)\,dx.$$

Hence Weyl's criterion holds for a characteristic function.

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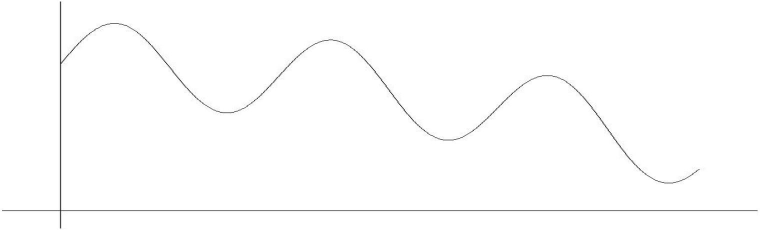

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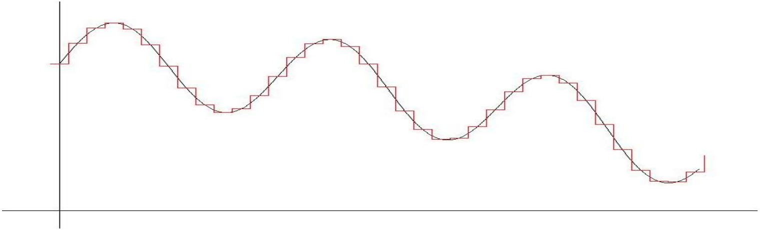

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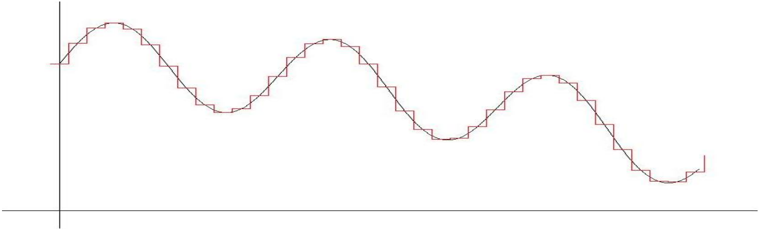

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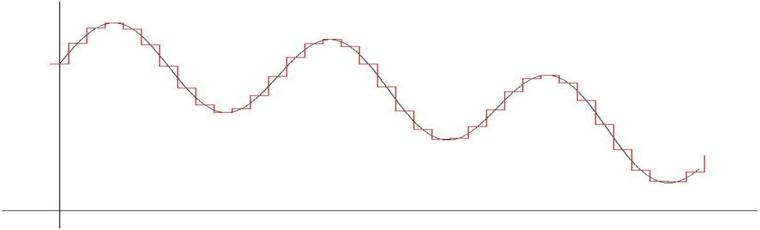

Hence (1) $\implies$ (2). (2) $\implies$ (3) is trivial (put $f(x) = e^{2\pi i \ell x}$).

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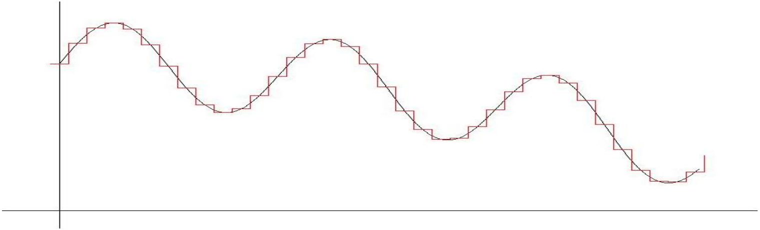

Hence (1) $\implies$ (2). (2) $\implies$ (3) is trivial (put $f(x) = e^{2\pi i \ell x}$).
(3) $\implies$ (1): Approximate $\chi_{[a,b]}$ by finite linear combinations of exponential functions. $\square$

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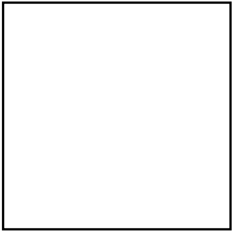

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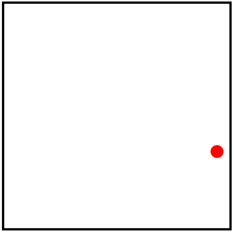

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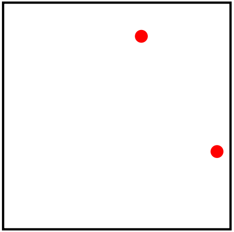

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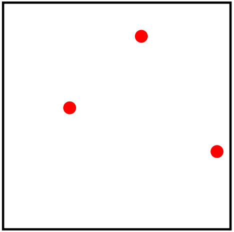

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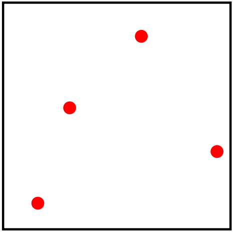

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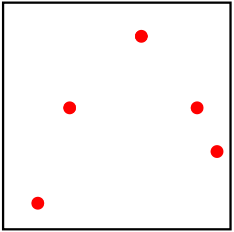

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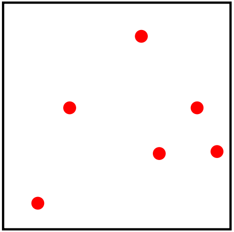

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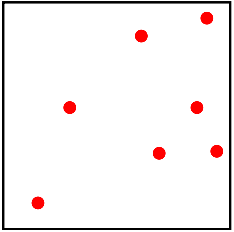

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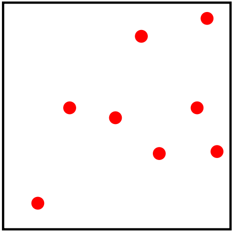

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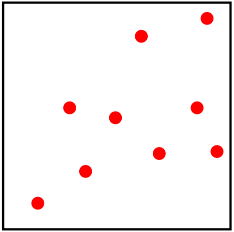

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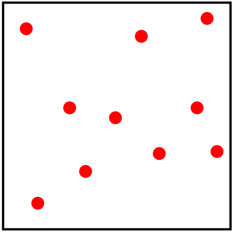

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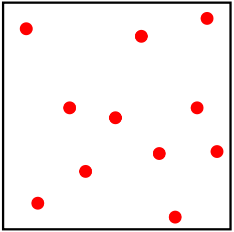

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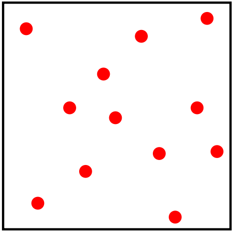

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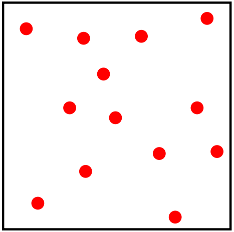

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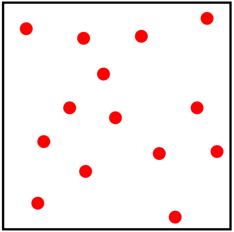

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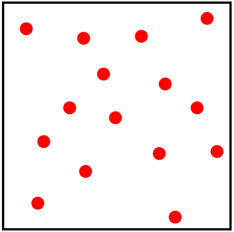

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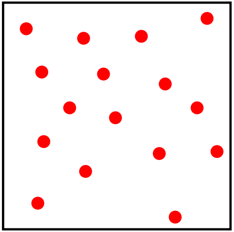

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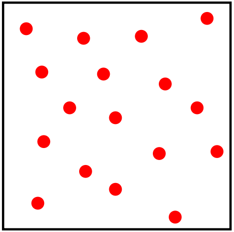

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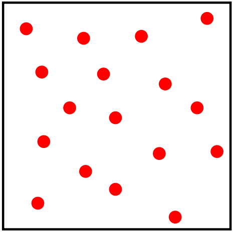

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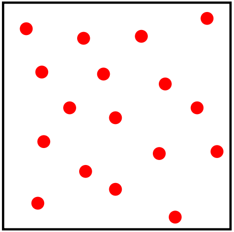

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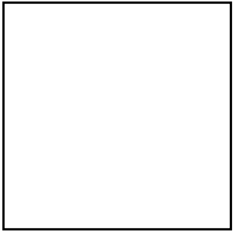

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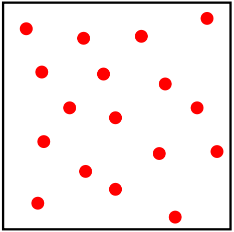

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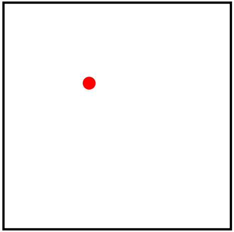

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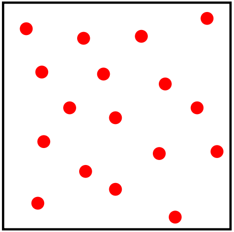

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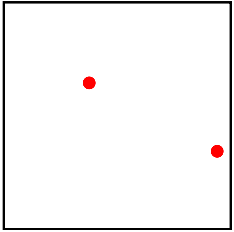

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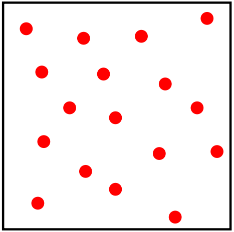

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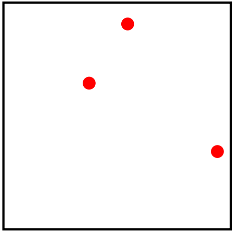

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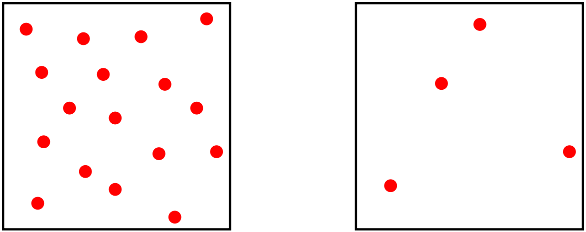

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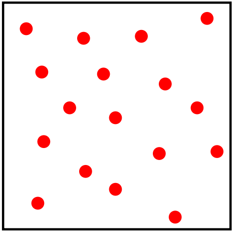

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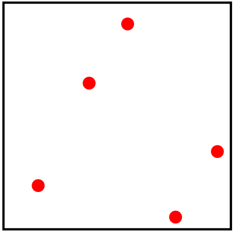

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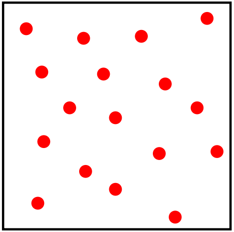

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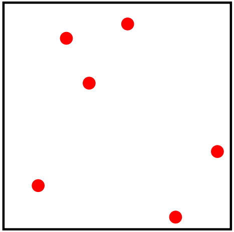

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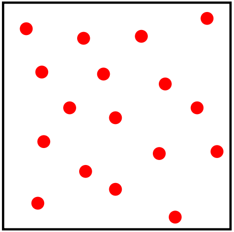

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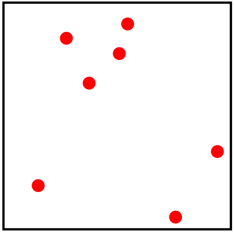

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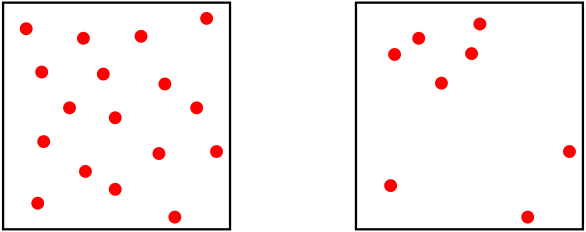

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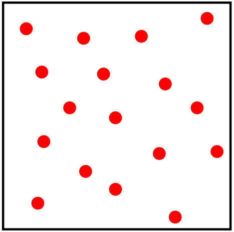

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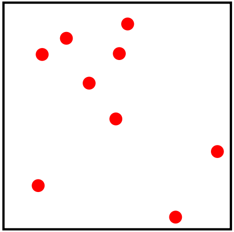

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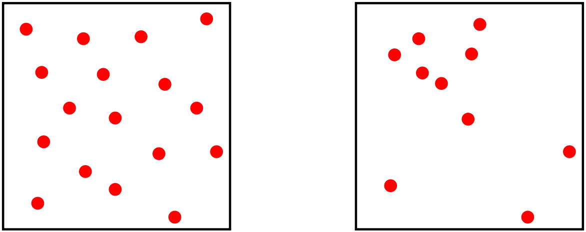

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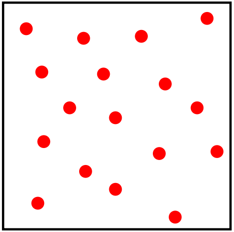

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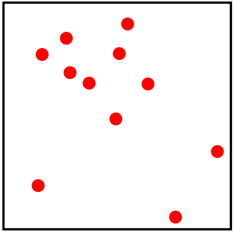

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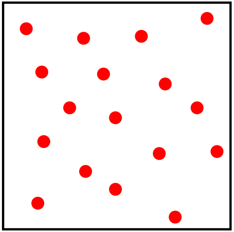

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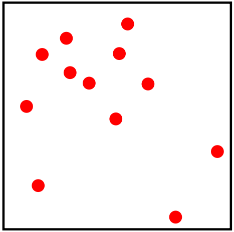

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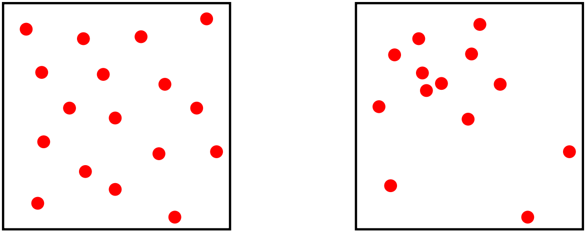

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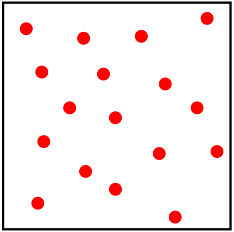

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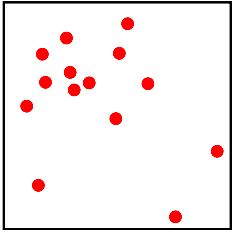

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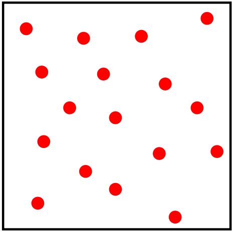

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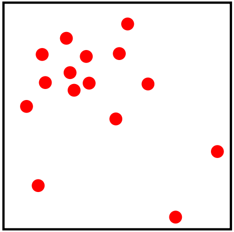

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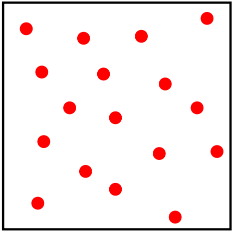

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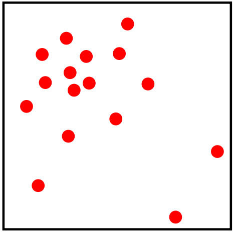

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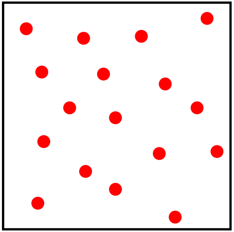

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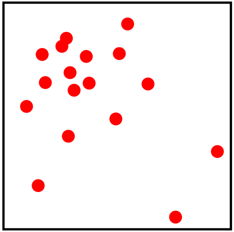

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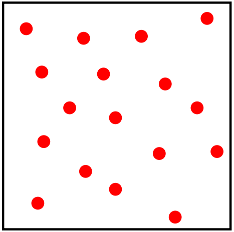

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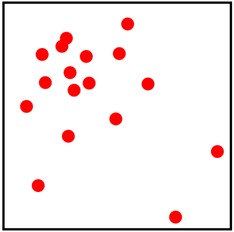

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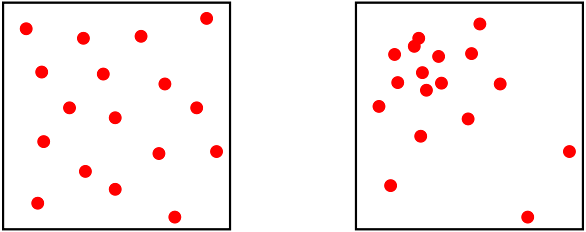

udm1 not udm

$x_n \in \mathbb{R}^k$ is udm1 if the frequency with which x_n mod 1 hits any k-dimensional cube equals the k-dimensional volume of that cube.

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We say $x_n = (x_n^{(1)}, \ldots, x_n^{(k)}) \in \mathbb{R}^k$ is udm1 if for all intervals $[a_1, b_1], \ldots, [a_k, b_k] \subset [0, 1]$ we have

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$$\frac{1}{n}\sum_{j=0}^{n-1} \exp(2\pi i(\ell_1 x_j^{(1)} + \cdots + \ell_k x_j^{(k)})) \longrightarrow 0$$

as $n \to \infty$.

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$$\begin{aligned} \alpha, 1 \text{ rationally indep.} \quad &\Leftrightarrow \quad \text{no rational solutions } r_1, r_2 \text{ to } r_1\alpha + r_2 = 0 \\ &\Leftrightarrow \quad \alpha \text{ irrational} \end{aligned}$$

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Theorem (Weyl)

Let $p(n) = \alpha_k n^k + \alpha_{k-1} n^{k-1} + \cdots + \alpha_1 n + \alpha_0$. If at least one of $\alpha_k, \ldots, \alpha_1$ is irrational then $p(n)$ is udm1.

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(In notes.) Idea:

$$\begin{aligned} |z_0 + z_1|^2 &= (z_0 + z_1)(\bar{z}_0 + \bar{z}_1) \\ &= |z_0|^2 + |z_1|^2 + z_0\bar{z}_1 + \bar{z}_0 z_1 \\ &= |z_0|^2 + |z_1|^2 + 2\,\mathrm{Re}\,(\bar{z}_0 z_1) \end{aligned}$$

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$$\Delta(k) \quad = \quad \text{“every polynomial } q \text{ of degree} \leq k \text{ with irrational leading coefficient is such that } q(n) \text{ is udm1.”}$$

$\Delta(1)$ is true as $\alpha n + \beta$ is udm1 if $\alpha \notin \mathbb{Q}$.
Suppose $\Delta(k-1)$ is true. We prove $\Delta(k)$ is true. Let $p(n) = \alpha_k n^k + \cdots + \alpha_1 n + \alpha_0$ be a degree k polynomial with irrational leading coefficient α_k.

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Lemma $\implies$ $p(n)$ is udm1. $\square$

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As m is arbitrary, the result follows. $\square$

Exponential Sequences

We now consider sequences of the form $x_n = \alpha^n, \alpha > 1$. This is a much harder and more delicate problem!

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Proof.

(Sketch) By Weyl's criterion, we want to show, for $\ell \in \mathbb{Z} \backslash \{0\}$,

$$A_n(x) = \left| \frac{1}{n} \sum_{j=0}^{n-1} e^{2\pi i \ell x_j} \right|^2 = \left| \frac{1}{n} \sum_{j=0}^{n-1} e^{2\pi i \ell \alpha^j x} \right|^2 \longrightarrow 0$$

for a.e. x.

For this it is sufficient to prove

$$\sum_n A_n(x) < \infty \text{ a.e.}$$

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Remark

We will give an easier proof that $\alpha^n x$ is udm1 for a.e. x when $\alpha \in \mathbb{Z}, \alpha > 1$ in a future lecture.

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No example of such an α *for which* α^n *is udm1 is known.*
Indeed, it is not even known if $(3/2)^n \mod 1$ *is dense!*

Next lecture

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We will define what is meant by an invariant measure, and study some invariant measures for the examples of dynamical systems that we saw in the first lecture.