MAGIC 010: Ergodic Theory

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Course Outline

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- There are 10 lectures.
- Each lecture has:
  - a set of slides,
  - a set of more detailed notes that contain more information about the material, details of arguments that are only sketched in the slides, and the exercises,
  - solutions to selected exercises.
- These will all be available via the Magic website.
- Assessment: take-home exam at the end of the course.

MAGIC: Ergodic Theory Lecture 1 - Examples of Dynamical Systems

January 24th 2013

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Let $T : X \to X$ be a dynamical system. Let $x \in X$. The *orbit* of x is the set $\{T^n x \mid n \geq 0\}$.

Example

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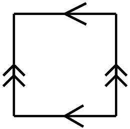
Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ denote the torus:



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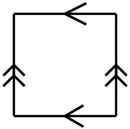
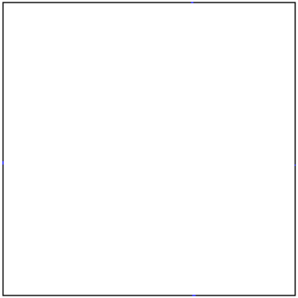
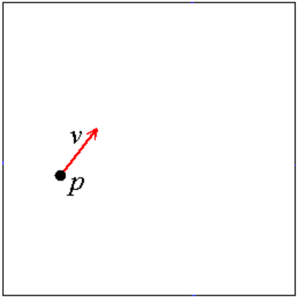


Let $X =$ the unit tangent bundle of $\mathbb{T}^2$. Then a point $x \in X$ is a pair $x = (p, v)$:

$$\begin{aligned} p &= \text{point on } \mathbb{T}^2 \\ v &= \text{unit vector at } v. \end{aligned}$$




v
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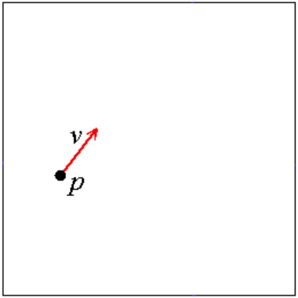
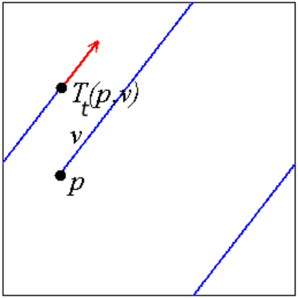


Then there is a unique straight line through p in the direction v. We can infinitely extend this line using the identifications. The dynamical system moves the point $x = (p, v)$ along this line at unit speed.



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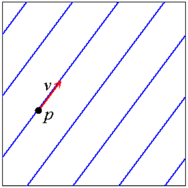

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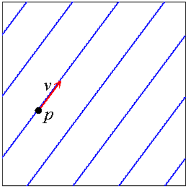

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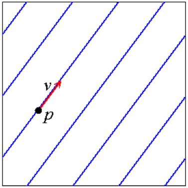

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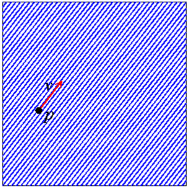
When projected to $\mathbb{T}^2$, some orbits are dense:



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$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) \longrightarrow \int f \, d\mu \qquad \mu\text{-a.e.}$$

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Note that both the circle and d-dimensional torus are Abelian groups under addition.

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1. If α is rational ($\alpha = \frac{p}{q}$, p, q coprime) then every orbit is periodic with period q.
2. If α is irrational then every orbit is dense.

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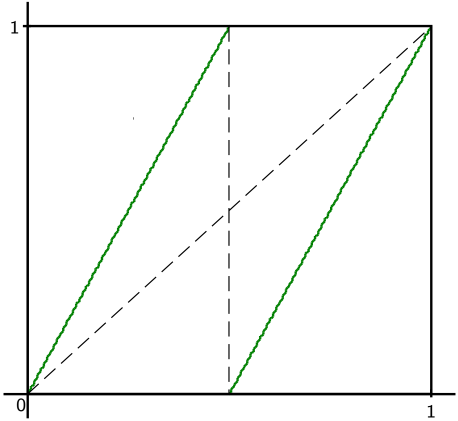

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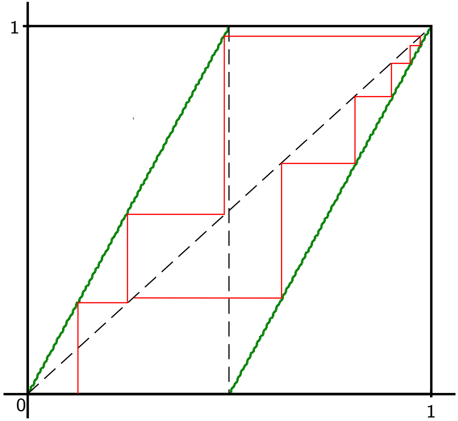

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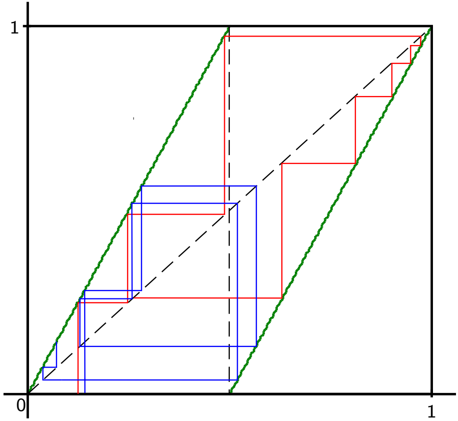

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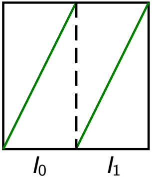

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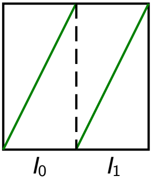

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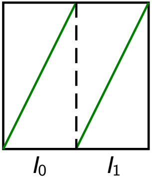

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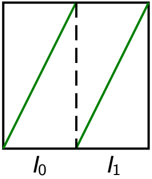

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We will return to shift maps and their uses later.

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$$T = T_A : X \to X : \quad x + \mathbb{Z}^k \mapsto Ax + \mathbb{Z}^k.$$

Endomorphisms and automorphisms of a torus

Let $X = \mathbb{R}^k/\mathbb{Z}^k$ be the k-dimensional torus.
Let $A = (a_{ij})$ be a $k \times k$ matrix with integer entries, $\det A \neq 0$.
Then we have a linear map

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We will often abuse notation by writing

$$T(x_1, \ldots, x_k) = A(x_1, \ldots, x_k) \bmod 1.$$

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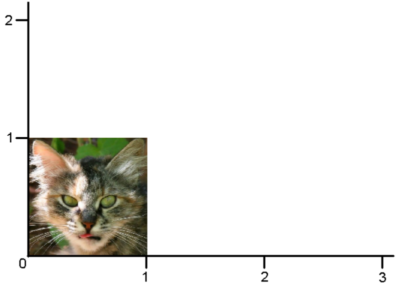

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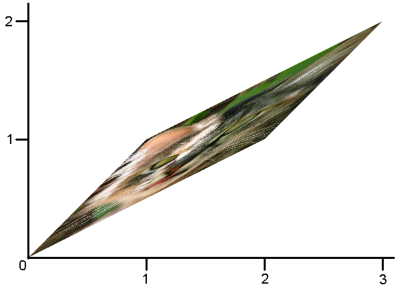

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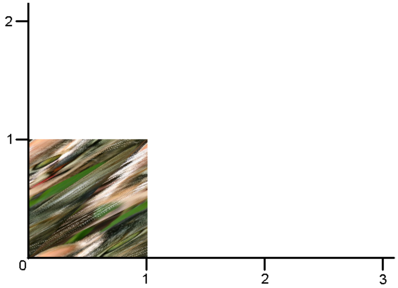

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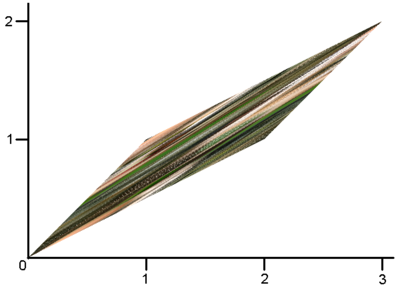

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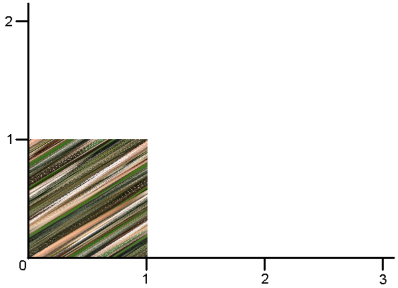

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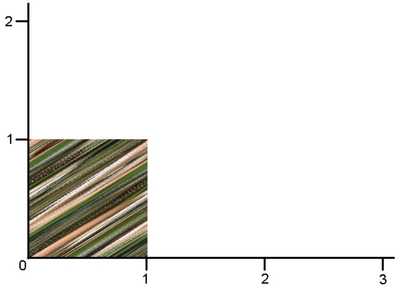

(CAT = **C**ontinuous **A**utomorphism of a **T**orus)

The Gauss Map

Every $x \in (0,1)$ can be expressed as a continued fraction

$$x = \cfrac{1}{x_0 + \cfrac{1}{x_1 + \cfrac{1}{x_2 + \dots}}} = [x_0, x_1, x_2, \dots].$$

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Facts about continued fractions:

- x has a finite continued fraction expansion iff x is rational.
- If x is irrational then it has a unique continued fraction expansion.

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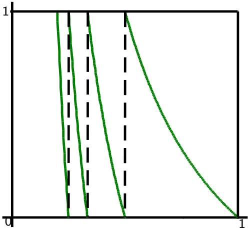

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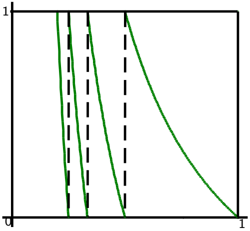

If x has continued fraction expansion $[x_0, x_1, x_2, \dots]$ then $T(x)$ has continued fraction expansion $[x_1, x_2, x_3, \dots]$.

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Define

$$\begin{aligned} A_{ij} = 1 &\quad \text{if symbol } j \text{ can follow symbol } i \\ A_{ij} = 0 &\quad \text{if symbol } j \text{ cannot follow symbol } i. \end{aligned}$$

Let

$$\begin{aligned}\Sigma_A^+ &= \{(x_j)_{j=0}^{\infty} \mid x_j \in S, A_{x_j x_{j+1}} = 1 \ \forall j \geq 0\} \\ &= \text{a } \textit{one-sided shift of finite type}\end{aligned}$$

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We need hypotheses on A to ensure that Σ_A has sufficient structure to be of interest.

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A is irreducible	$\iff$	For all vertices i, j, there exists a path in Γ (with length depending on i, j) from i to j.
A is aperiodic	$\iff$	as above, but the length of the path can be chosen to be independent of i, j.

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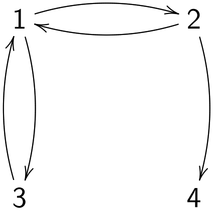

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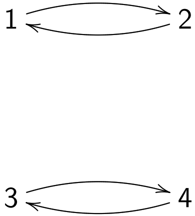

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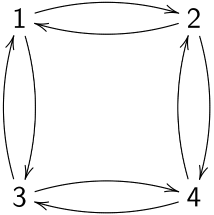

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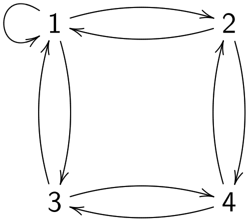

aperiodic

A topology on Σ_A, Σ_A^+

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Let $x = (x_j)_{j=0}^{\infty}$, $y = (y_j)_{j=0}^{\infty} \in \Sigma_A^+$. Fix any $\theta \in (0,1)$.
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$$n(x,y) = \begin{cases} \sup\{n \mid x_j = y_j,\ j = 0,1,\ldots,n-1\} & \text{if } x \neq y; \\ \infty & \text{if } x = y. \end{cases} \qquad (*)$$

Define $d_\theta(x,y) = \theta^{n(x,y)}$. Then d_θ is a metric on Σ_A^+. (There is a similar formula for the two-sided case - replace $j \in \{0,1\ldots,n\}$ by $j \in \{-n,\ldots,0,\ldots,n\}$ in $(*)$.)

Remark

The exact choice of θ is, for the moment, unimportant: any $\theta \in (0,1)$ gives the same topology.

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The shift map is chaotic:

- sensitive dependence on initial conditions
- the periodic points are dense
- there is a dense orbit.

(See exercises.)

Markov partitions and symbolic coding

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For example we have already seen that the doubling map can be coded by the full one-sided 2-shift:

$$\begin{tikzcd} \Sigma_2 \arrow[r, "\sigma"] \arrow[d, "\pi"'] & \Sigma_2 \arrow[d, "\pi"] \\ \mathbb{R}/\mathbb{Z} \arrow[r, "T"'] & \mathbb{R}/\mathbb{Z} \end{tikzcd}$$

$Tx = 2x \mod 1$ $\qquad\qquad$ $\pi(x_j) = \sum_{j=0}^{\infty} \frac{x_j}{2^{j+1}}$

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Definition

A linear toral automorphism $T = T_A$ is *hyperbolic* if A has no eigenvalues of modulus 1.

Example

The Cat map $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is hyperbolic.

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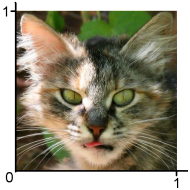

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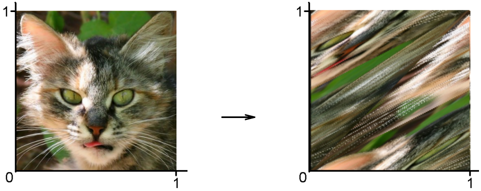

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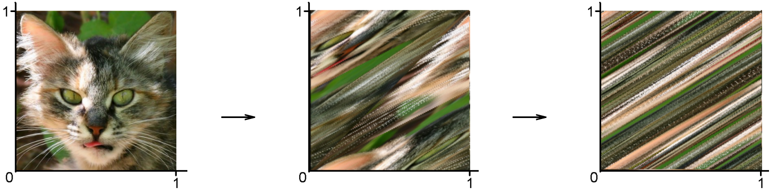

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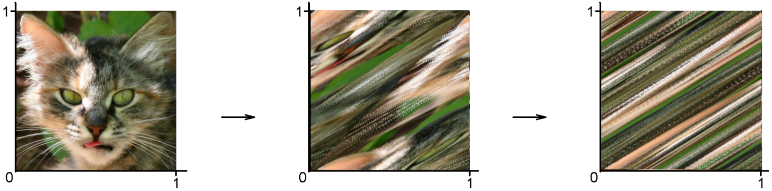

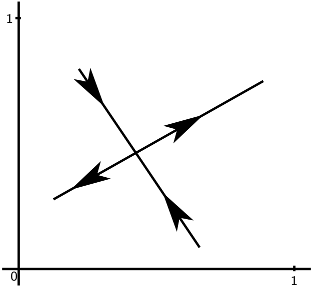

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Remark

It follows that T is chaotic.

One can use symbolic dynamics to prove many other results about hyperbolic dynamical systems.

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We will also see some applications to number theory.