2. Uniform distribution mod 1

§2.1 Introduction

Recall that ergodicity is concerned with how a typical orbit of a dynamical system is distributed throughout the phase space. In this lecture we study an apparently simpler problem, namely how the fractional parts of sequences of reals are distributed in the unit interval. We also discuss some applications of a number-theoretic nature.

§2.2 Uniform distribution and Weyl's criterion in one dimension

Let x_n be a sequence of real numbers. We may decompose x_n as the sum of its *integer part* $[x_n] = \sup\{m \in \mathbb{Z} \mid m \leq x_n\}$ (i.e. the largest integer which is less than or equal to x_n) and its *fractional part* $\{x_n\} = x_n - [x_n]$. Clearly, $0 \leq \{x_n\} < 1$. The study of x_n mod 1 is the study of the sequence $\{x_n\}$ in $[0, 1)$.

Definition. We say that the sequence x_n is *uniformly distributed mod 1* if for every a, b with $0 \leq a < b < 1$, we have that

$$\frac{1}{n}\mathrm{card}\{j \mid 0 \leq j \leq n-1,\ \{x_j\} \in [a, b]\} \to b - a, \quad \text{as } n \to \infty.$$

(The condition is saying that the frequency with which the sequence $\{x_n\}$ lies in $[a, b]$ converges to $b - a$, the length of the interval.)

Remark We can replace $[a, b]$ by $[a, b)$, $(a, b]$ or (a, b) without changing the definition.

The following result gives a necessary and sufficient condition for x_n to be uniformly distributed mod 1.

Theorem 2.1 (Weyl's Criterion)
The following are equivalent:

(i) *the sequence x_n is uniformly distributed mod 1;*

(ii) *for any continuous function $f : [0, 1] \to \mathbb{R}$ with $f(0) = f(1)$ we have*

$$\frac{1}{n}\sum_{j=0}^{n-1} f(\{x_j\}) \to \int_0^1 f(x)\,dx; \tag{2.1}$$

(iii) for each $\ell \in \mathbb{Z} \setminus \{0\}$, we have

$$\frac{1}{n}\sum_{j=0}^{n-1} e^{2\pi i \ell x_j} \to 0$$

as $n \to \infty$.

Remarks.

1. As a (pedantic) grammatical point, criterion is singular and criteria is plural. Weyl's criterion is that conditions (i) and (iii) are equivalent. Condition (ii) is stated explicitly as it bears a close resemblance to an ergodic theorem.

2. Condition (ii) can be replaced by (ii'): that the convergence in (2.1) holds for all Riemann integrable functions $f : [0,1] \to \mathbb{R}$ with $f(0) = f(1)$. In the following proof we assume some familiarity with properties of the Riemann integral.

3. Condition (ii) suggests the following extension of the definition of uniform distribution mod 1. Let μ be a Borel probability measure on $[0,1]$. Then the sequence $x_n \in \mathbb{R}$ is *μ-uniformly distributed mod 1* if for any continuous function $f : [0,1] \to \mathbb{R}$ with $f(0) = f(1)$ we have

$$\frac{1}{n}\sum_{j=0}^{n-1} f(\{x_j\}) \to \int f \, d\mu.$$

4. The assumption that $f(0) = f(1)$ in (ii) and (ii') can also be removed.

Proof. For notational simplicity, and since $e^{2\pi i x_j} = e^{2\pi i \{x_j\}}$, we assume without loss of generality that $x_j = \{x_j\}$.

We prove (i) implies (ii). Suppose that x_j is uniformly distributed mod 1. If $\chi_{[a,b]}$ is the characteristic function of the interval $[a,b]$, then we may rewrite the definition of uniform distribution as

$$\frac{1}{n}\sum_{j=0}^{n-1} \chi_{[a,b]}(x_j) \to \int_0^1 \chi_{[a,b]}(x)\, dx, \quad \text{as } n \to \infty.$$

From this we deduce that

$$\frac{1}{n}\sum_{j=0}^{n-1} g(x_j) \to \int_0^1 g(x)\, dx, \quad \text{as } n \to \infty,$$

whenever g is a step function, i.e., a finite linear combination of characteristic functions of intervals.

Now let f be a continuous function on $[0,1]$. Then, given $\varepsilon > 0$, we can find a step function g with $\|f-g\|_\infty \le \varepsilon$. We have the estimate

$$\begin{aligned}
&\left|\frac{1}{n}\sum_{j=0}^{n-1} f(x_j) - \int_0^1 f(x)\,dx\right| \\
&\quad\le \left|\frac{1}{n}\sum_{j=0}^{n-1}(f(x_j)-g(x_j))\right| + \left|\frac{1}{n}\sum_{j=0}^{n-1} g(x_j) - \int_0^1 g(x)\,dx\right| \\
&\qquad + \left|\int_0^1 g(x)\,dx - \int_0^1 f(x)\,dx\right| \\
&\quad\le 2\varepsilon + \left|\frac{1}{n}\sum_{i=0}^{n-1} g(x_j) - \int_0^1 g(x)\,dx\right|.
\end{aligned}$$

Since the last term converges to zero as $n \to \infty$, we obtain

$$\limsup_{n\to\infty}\left|\frac{1}{n}\sum_{j=0}^{n-1} f(x_j) - \int_0^1 f(x)\,dx\right| \le 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this gives us that

$$\frac{1}{n}\sum_{j=0}^{n-1} f(x_j) \to \int_0^1 f(x)\,dx,$$

as $n \to \infty$.

Condition (ii) trivially implies (iii) by setting $f(x) = e^{2\pi i\ell x}$, for each $\ell \in \mathbb{Z}, \ell \neq 0$.

We prove (iii) implies (i). Suppose that (iii) holds. Then

$$\frac{1}{n}\sum_{j=0}^{n-1} g(x_j) \to \int_0^1 g(x)\,dx, \quad \text{as } n \to \infty,$$

whenever $g(x) = \sum_{k=1}^m \alpha_k e^{2\pi i\ell_k x}$ is a trigonometric polynomial, i.e. a finite linear combination of exponential functions.

Let f be any continuous function on $[0,1]$ with $f(0) = f(1)$. Given $\varepsilon > 0$ we can find a trigonometric polynomial g such that $\|f-g\|_\infty \le \varepsilon$. As in the first part of the proof, we can conclude that

$$\frac{1}{n}\sum_{j=0}^{n-1} f(x_j) \to \int_0^1 f(x)\,dx, \quad \text{as } n \to \infty.$$

Now consider the interval $[a,b] \subset [0,1)$. Given $\varepsilon > 0$, we can find continuous functions f_1, f_2 (with $f_1(0) = f_1(1)$, $f_2(0) = f_2(1)$) such that

$$f_1 \le \chi_{[a,b]} \le f_2$$

and

$$\int_0^1 f_2(x) - f_1(x)\, dx \le \varepsilon.$$

We then have that

$$\begin{aligned}
\liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \chi_{[a,b]}(x_j) &\ge \liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} f_1(x_j) = \int_0^1 f_1(x)\, dx \\
&\ge \int_0^1 f_2(x)\, dx - \varepsilon \ge \int_0^1 \chi_{[a,b]}(x)\, dx - \varepsilon
\end{aligned}$$

and

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \chi_{[a,b]}(x_j) &\le \limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} f_2(x_j) = \int_0^1 f_2(x)\, dx \\
&\le \int_0^1 f_1(x)\, dx + \varepsilon \le \int_0^1 \chi_{[a,b]}(x)\, dx + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have shown that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \chi_{[a,b]}(x_j) = \int_0^1 \chi_{[a,b]}(x)\, dx = b - a,$$

so that x_i is uniformly distributed mod 1. □

§2.2.1 Example: the sequence $\mathbf{x_n} = \alpha\mathbf{n}$

The behaviour of the sequence $x_n = \alpha n$ depends on whether α is rational or irrational. If $\alpha \in \mathbb{Q}$, it is easy to see that $\{\alpha n\}$ can take on only finitely many values in $[0, 1)$: if $\alpha = p/q$ ($p \in \mathbb{Z}$, $q \in \mathbb{N}$, $\mathrm{hcf}(p, q) = 1$) then $\{\alpha n\}$ takes the q distinct values

$$0, \left\{\frac{p}{q}\right\}, \left\{\frac{2p}{q}\right\}, \ldots, \left\{\frac{(q-1)p}{q}\right\}.$$

In particular, $\{\alpha n\}$ is not uniformly distributed mod 1.

If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ then the situation is completely different. We shall apply Weyl's Criterion. For $\ell \in \mathbb{Z} \setminus \{0\}$, $e^{2\pi i\ell\alpha} \ne 1$, so we have

$$\frac{1}{n}\sum_{j=0}^{n-1} e^{2\pi i\ell\alpha j} = \frac{1}{n}\frac{e^{2\pi i\ell\alpha n} - 1}{e^{2\pi i\ell\alpha} - 1}.$$

Hence

$$\left|\frac{1}{n}\sum_{j=0}^{n-1} e^{2\pi i\ell\alpha j}\right| \le \frac{1}{n}\frac{2}{|e^{2\pi i\ell\alpha} - 1|} \to 0, \quad \text{as } n \to \infty.$$

Hence αn is uniformly distributed mod 1.

Remark More generally, we could consider the sequence $x_n = \alpha n + \beta$. It is easy to see by modifying the above arguments that x_n is uniformly distributed mod 1 if and only if α is irrational. Thus we have a result about the uniform distribution of the values of a linear polynomial in one dimension. We will generalise this result below.

§2.3 Uniform distribution mod 1 in higher dimensions

We shall now look at the uniform distribution of sequences in $\mathbb{R}^k$. We will say that a sequence $x_n = (x_n^1, \ldots, x_n^k) \in \mathbb{R}^k$ is uniformly distributed mod 1 if, given any k-dimensional cube, the frequency with which the fractional parts of x_n lie in the cube is equal to its k-dimensional volume of the cube.

Definition. A sequence $x_n = (x_n^1, \ldots, x_n^k) \in \mathbb{R}^k$ is said to be *uniformly distributed mod 1* if, for each choice of k intervals $[a_1, b_1], \ldots, [a_k, b_k] \subset [0, 1)$, we have that

$$\frac{1}{n}\sum_{j=0}^{n-1}\prod_{i=1}^{k}\chi_{[a_i,b_i]}(\{x_j^i\}) \to \prod_{i=1}^{k}(b_i - a_i), \quad \text{as } n \to \infty.$$

We have the following criterion for uniform distribution.

Theorem 2.2 (Multi-dimensional Weyl's Criterion)
Let $x_n = (x_n^{(1)}, \ldots, x_n^{(k)}) \in \mathbb{R}^k$. The following are equivalent:

(i) the sequence $x_n \in \mathbb{R}^k$ is uniformly distributed mod 1;

(ii) for any continuous function $f : \mathbb{R}^k/\mathbb{Z}^k \to \mathbb{R}$ we have

$$\frac{1}{n}\sum_{j=0}^{n-1} f(\{x_j^{(1)}\}, \ldots, \{x_j^{(k)}\}) \to \int \cdots \int f(x_1, \ldots, x_k)\, dx_1 \ldots dx_k;$$

(iii) for all $\ell = (\ell_1, \ldots, \ell_k) \in \mathbb{Z}^k \setminus \{0\}$ we have

$$\frac{1}{n}\sum_{j=0}^{n-1} e^{2\pi i(\ell_1 x_j^{(1)} + \cdots + \ell_k x_j^{(k)})} \to 0$$

as $n \to \infty$.

Remark Here and throughout $0 \in \mathbb{Z}^k$ denotes the zero vector $(0, \ldots, 0)$.

Proof. The proof is essentially the same as in the case $k = 1$. □

§2.3.1 Example: the sequence $\mathbf{x_n} = (\alpha_1 \mathbf{n}, \ldots, \alpha_k \mathbf{n})$

We shall apply Theorem 2.2 to the sequence $x_n = (\alpha_1 n, \ldots, \alpha_k n)$, for real numbers $\alpha_1, \ldots, \alpha_k$.

Definition. Real numbers $v_1, \ldots, v_s \in \mathbb{R}$ are said to be *rationally independent* if the only rationals $r_1, \ldots, r_s \in \mathbb{Q}$ such that

$$r_1 v_1 + \cdots + r_s v_s = 0$$

are $r_1 = \cdots = r_s = 0$.

Proposition 2.3
Let $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$. Then the following are equivalent:

(i) the sequence $x_n = (\alpha_1 n, \ldots, \alpha_k n) \in \mathbb{R}^k$ is uniformly distributed mod 1;

(ii) $\alpha_1, \ldots, \alpha_k$ and 1 are rationally independent.

Proof. The proof is similar to the discussion in §2.2.1 and we leave it as an exercise. □

§2.4 Weyl's theorem on polynomials

We have seen that $n\alpha + \beta$ is uniformly distributed mod 1 if α is irrational. Weyl's theorem generalises this to polynomials of higher degree. Write

$$p(n) = \alpha_k n^k + \alpha_{k-1} n^{k-1} + \cdots + \alpha_1 n + \alpha_0.$$

Theorem 2.4 (Weyl)
If any one of $\alpha_1, \ldots, \alpha_k$ is irrational then $p(n)$ is uniformly distributed mod 1.

To prove this theorem we shall need the following technical result.

Lemma 2.5 (van der Corput's Inequality)
Let $z_0, \ldots, z_{n-1} \in \mathbb{C}$ and let $1 < m < n$. Then

$$\begin{aligned} m^2 \left| \sum_{j=0}^{n-1} z_j \right|^2 \quad &\leq \quad m(n+m) \sum_{j=0}^{n-1} |z_j|^2 \\ &\quad + 2(n+m) \operatorname{Re} \sum_{j=1}^{m-1} (m-j) \sum_{i=0}^{n-1-j} z_{i+j} \bar{z}_i. \end{aligned}$$

Proof. The proof is essentially an exercise in multiplying out a product and some careful book-keeping of the cross-terms.

Consider the following sums:

$$
\begin{array}{ccccccccccccc}
s_0 & = & z_0 & & & & & & & & & & \\
s_1 & = & z_0 & + & z_1 & & & & & & & & \\
\vdots & & \vdots & & \vdots & & \ddots & & & & & & \\
s_{m-1} & = & z_0 & + & z_1 & + & \cdots & + & z_{m-1} & & & & \\
s_m & = & & & z_1 & + & \cdots & + & z_{m-1} & + & z_m & & \\
\vdots & & & & & \ddots & & & & & & \ddots & \\
s_{n-1} & = & & & & & z_{n-m} & + & & & \cdots & + & z_{n-1} \\
s_n & = & & & & & & & z_{n-m+1} & + & \cdots & + & z_{n-1} \\
\vdots & & & & & & & & & & & \ddots & \vdots \\
s_{n+m-1} & = & & & & & & & & & & & z_{n-1}
\end{array}
$$

Then each z_i occurs in exactly m of the sums s_j. Hence

$$s_0 + \cdots + s_{n+m-1} = m(z_0 + \cdots + z_{n-1})$$

so that

$$
\begin{aligned}
m^2 \left| \sum_{j=0}^{n-1} z_j \right| &= |s_0 + \cdots + s_{n+m-1}|^2 \\
&\le (|s_0| + \cdots + |s_{n+m-1}|)^2 \\
&\le (n+m)(|s_0|^2 + \cdots + |s_{n+m-1}|^2),
\end{aligned}
$$

where the final inequality follows from the $(n+m)$-dimensional Cauchy-Schwarz inequality.

Recall that $|s_j|^2 = s_j \bar{s_j}$. Expanding out this product and recalling that $2\,\mathrm{Re}(z) = z + \bar{z}$ we have that

$$|s_j|^2 = \sum_k |z_k|^2 + 2\,\mathrm{Re} \sum_{k,l} z_k \bar{z}_l$$

where the first sum is over all indices k of the z_i occurring in the definition of s_j, and the second sum is over the indices $l < k$ of the z_i occurring in the definition of s_j.

Noting that the the number of time the term $z_k \bar{z}_l$ occurs in $|s_0|^2 + \cdots + |s_{n+m-1}|^2$ is equal to $m - (l-k)$, we can write

$$|s_0|^2 + \cdots + |s_{n+m-1}|^2 \le m \sum_{j=0}^{n-1} |z_j|^2 + 2\,\mathrm{Re} \sum_{j=1}^{m-1} (m-j) \sum_{i=0}^{n-1-j} z_{i+j} \bar{z}_i$$

and the result follows. $\square$

Let $x_n \in \mathbb{R}$. For each $m \geq 1$ define the sequence $x_n^{(m)} = x_{n+m} - x_n$ of m^{th} differences. The following lemma allows us to infer the uniform distribution of the sequence x_n if we know the uniform distribution of the each of the m^{th} differences of x_n.

Lemma 2.6
Let $x_n \in \mathbb{R}$ be a sequence. Suppose that for each $m \geq 1$ the sequence $x_n^{(m)}$ of m^{th} differences is uniformly distributed mod 1. Then x_n is uniformly distributed mod 1.

Proof. We shall apply Weyl's Criterion. We need to show that if $\ell \in \mathbb{Z} \backslash \{0\}$ then

$$\frac{1}{n} \sum_{j=0}^{n-1} e^{2\pi i \ell x_j} \to 0, \quad \text{as } n \to \infty.$$

Let $z_j = e^{2\pi i \ell x_j}$ for $j = 0, \ldots, n-1$. Note that $|z_j| = 1$. Let $1 < m < n$. By van der Corput's inequality,

$$\begin{aligned}
\frac{m^2}{n^2} \left| \sum_{j=0}^{n-1} e^{2\pi i \ell x_j} \right|^2 &\leq \frac{m}{n^2}(n+m)n \\
&\quad + \frac{2(n+m)}{n} \operatorname{Re} \sum_{j=1}^{m-1} \frac{(m-j)}{n} \sum_{i=0}^{n-1-j} e^{2\pi i \ell (x_{i+j} - x_i)} \\
&= \frac{m}{n}(m+n) + \frac{2(n+m)}{n} \operatorname{Re} \sum_{j=1}^{m-1} (m-j) A_{n,j}
\end{aligned}$$

where

$$A_{n,j} = \frac{1}{n} \sum_{i=0}^{n-1-j} e^{2\pi i \ell (x_{i+j} - x_i)} = \frac{1}{n} \sum_{i=0}^{n-1-j} e^{2\pi i \ell x_i^{(j)}}.$$

As the sequence $x_i^{(j)}$ of j^{th} differences is uniformly distributed mod 1, by Weyl's criterion we have that $A_{n,j} \to 0$ for each $j = 1, \ldots, m-1$. Hence for each $m \geq 1$

$$\limsup_{n \to \infty} \frac{m^2}{n^2} \left| \sum_{j=0}^{n-1} e^{2\pi i \ell x_j} \right|^2 \leq \limsup_{n \to \infty} m \frac{(n+m)}{n} = m.$$

Hence, for each $m > 1$ we have

$$\limsup_{n \to \infty} \frac{1}{n} \left| \sum_{j=0}^{n-1} e^{2\pi i \ell x_j} \right| \leq \frac{1}{\sqrt{m}}.$$

As $m > 1$ is arbitrary, the result follows. □

Proof of Weyl's Theorem. We will only prove Weyl's theorem in the special case where the leading coefficient α_k of

$$p(n) = \alpha_k n^k + \cdots + \alpha_1 n + \alpha_0$$

is irrational. (The general case, where α_i is irrational for some $1 \leq i \leq k$ can be deduced very easily from this special case, and we leave this as an exercise.)

We shall use induction on the degree of p. Let $\Delta(k)$ denote the statement 'for every polynomial p of degree $\leq k$, with irrational leading coefficient, the sequence $p(n)$ is uniformly distributed mod 1'. We know that $\Delta(1)$ is true.

Suppose that $\Delta(k-1)$ is true. Let $p(n) = \alpha_k n^k + \cdots + \alpha_1 n + \alpha_0$ be any polynomial of degree k with α_k irrational. For each $m \in \mathbb{N}$, we have that

$$\begin{aligned}
&p(n+m) - p(n) \\
&\quad = \alpha_k (n+m)^k + \alpha_{k-1}(n+m)^{k-1} + \cdots + \alpha_1(n+m) + \alpha_0 \\
&\qquad - \alpha_k n^k - \alpha_{k-1} n^{k-1} - \cdots - \alpha_1 n - \alpha_0 \\
&\quad = \alpha_k n^k + \alpha_k k n^{k-1} m + \cdots + \alpha_{k-1} n^{k-1} + \alpha_{k-1}(k-1) n^{k-2} h \\
&\qquad + \cdots + \alpha_1 n + \alpha_1 m + \alpha_0 - \alpha_k n^k - \alpha_{k-1} n^{k-1} - \cdots - \alpha_1 n - \alpha_0.
\end{aligned}$$

After cancellation, we can see that, for each m, $p(n+m) - p(n)$ is a polynomial of degree $k-1$, with irrational leading coefficient $\alpha_k k m$. Therefore, by the inductive hypothesis, $p(n+m) - p(n)$ is uniformly distributed mod 1. We may now apply Lemma 2.6 to conclude that $p(n)$ is uniformly distributed mod 1 and so $\Delta(k)$ holds. This completes the induction. □

§2.5 The sequence $\mathbf{x_n} = \alpha^{\mathbf{n}}\mathbf{x}$

Having considered polynomial functions, we now consider exponential functions. We would like to take $\alpha > 1$ and to study the uniform distribution mod 1 of the sequence α^n. This is a hard, and in general, still unsolved problem. We will consider the sequence $\alpha^n x$ where x is some fixed parameter. We will show that for 'typical' points $x \in \mathbb{R}$, the sequence $\alpha^n x$ is uniformly distributed mod 1. 'Typical' in this context means for Lebesgue almost every point $x \in \mathbb{R}$.

Theorem 2.7
Let $\alpha > 1$. Then for Lebesgue-almost every $x \in \mathbb{R}$, the sequence $x_n = \alpha^n x$ is uniformly distributed mod 1.

Remarks.

1. The proof uses the observation that if $\int f^2(x)\,dx = 0$ then $f = 0$ almost everywhere. If we think of x in this integral as a parameter, then this allows us to conclude that $f = 0$ for almost every value of the parameter x.

2. We will give an alternative proof of this theorem in the case $\alpha \in \mathbb{Z}$ later.

The following lemma says, essentially, that in order to verify Weyl's criterion, it is sufficient to consider the limit in Theorem 2.1(iii) as $n \to \infty$ through any one of a particular class of subsequence.

Lemma 2.8
Suppose that $n_k \in \mathbb{N}$ is an increasing sequence such that $n_{k+1}/n_k \to 1$ as $n_k \to \infty$. Suppose that

$$\lim_{n_j \to \infty} \frac{1}{n_k} \sum_{j=0}^{n_k - 1} e^{2\pi i \ell x_j} = 0.$$

Then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} e^{2\pi i \ell x_j} = 0.$$

Proof. Given $n \in \mathbb{N}$, choose $k = k(n)$ such that $n_k \le n < n_{k+1}$. As $n_{k+1}/n_k \to 1$ as $n_k \to \infty$, it follows that $n/n_k \to 1$ as $n \to \infty$. Hence

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n} \left| \sum_{j=0}^{n-1} e^{2\pi i \ell x_j} \right| &\le \limsup_{n\to\infty} \frac{n_k}{n} \left(\frac{1}{n_k} \left| \sum_{j=0}^{n_k-1} e^{2\pi i \ell x_j} \right| + \frac{1}{n_k} \sum_{j=n_k}^{n-1} \left| e^{2\pi i \ell x_j} \right| \right) \\
&\le \limsup_{n\to\infty} \frac{n_k}{n} \left(\frac{1}{n_k} \left| \sum_{j=0}^{n_k-1} e^{2\pi i \ell x_j} \right| + \frac{n-n_k}{n_k} \right) \\
&= 0
\end{aligned}$$

as $(n - n_k)/n_k \le (n_{k+1}/n_k) - 1 \to 0$. $\square$

Proof of Theorem 2.7. Let $[a, b] \subset \mathbb{R}$ be an arbitrary interval, with $a < b$. Let $\ell \in \mathbb{Z} \setminus \{0\}$. Let

$$A_n(x) = \frac{1}{n} \sum_{j=0}^{n-1} e^{2\pi i \ell \alpha^j x}.$$

We want to show that $A_n(x) \to 0$ for a.e. $x \in [a, b]$. We will do this by showing that for a.e. $x \in [a, b]$ we have that $A_{n_k} \to 0$ along a subsequence with the properties given in Lemma 2.8; indeed, we will choose $n_k = k^2$.

To show $A_{n_k}(x) \to 0$ for a.e. $x \in [a, b]$, it is sufficient to prove that $\sum_k |A_{n_k}(x)|^2 < \infty$ for a.e. $x \in [a, b]$, as the summands in a convergent series tend to zero. Let

$$I_n = \int_a^b |A_n(x)|^2 \, dx.$$

As $|A_n(x)|^2 \geq 0$, by Tonelli's theorem we can interchange the order of integration and summation to obtain

$$\int_a^b \sum_k |A_{n_k}(x)|^2 \, dx = \sum_k \int_a^b |A_{n_k}(x)^2 \, dx = \sum_k I_{n_k}.$$

Hence, by the remark preceding the proof, if we can show that $\sum_k I_{n_k} < \infty$ then it follows that $\sum_k |A_{n_k}(x)|^2 < \infty$ for a.e. $x \in [a, b]$.

Now

$$\begin{aligned} I_n &= \int_a^b \left| \frac{1}{n} \sum_{j=0}^{n-1} e^{2\pi i \ell \alpha^j x} \right|^2 dx \\ &= \frac{1}{n^2} \sum_{k,l=0}^{n-1} \int_a^b e^{2\pi i \ell (\alpha^k - \alpha^l) x} \, dx. \end{aligned} \tag{2.2}$$

We can estimate the summand in (2.2). First note that for $k = l$ the integral is equal to $b - a$. For $k > l$ note that

$$\begin{aligned} \left| \int_a^b e^{2\pi i \ell (\alpha^k - \alpha^l) x} \, dx \right| &= \frac{1}{2\pi(\alpha^k - \alpha^l)} \left| e^{2\pi i \ell (\alpha^k - \alpha^l) b} - e^{2\pi i \ell (\alpha^k - \alpha^l) a} \right| \\ &\leq \frac{1}{\pi(\alpha^k - \alpha^l)}. \end{aligned}$$

Similarly, for $k < l$ we have (recalling that $\alpha > 1$)

$$\left| \int_a^b e^{2\pi i \ell (\alpha^k - \alpha^l) x} \, dx \right| \leq \frac{1}{\pi(\alpha^k - \alpha^l)}.$$

If $k > l$ then

$$\alpha^k - \alpha^l = (\alpha^{k-1} + \cdots + \alpha^l)(\alpha - 1) \geq (k - l)(\alpha - 1).$$

Hence

$$\begin{aligned} |I_n| &\leq \frac{(b-a)}{n^2} n + \frac{2}{n^2} \sum_{k=1}^{n-1} \sum_{l=0}^{k-1} \frac{1}{\pi(\alpha^k - \alpha^l)} \\ &\leq \frac{b-a}{n} + \frac{2(\alpha - 1)}{\pi n^2} \sum_{k=1}^{n-1} \sum_{l=0}^{k-1} \frac{1}{k - l}. \end{aligned}$$

Now

$$\sum_{l=0}^{k-1} \frac{1}{k-l} = \sum_{l=1}^{k} \frac{1}{l} \leq C \log k \leq C \log n$$

for some constant $C > 0$. Hence

$$|I_n| \leq \frac{b-a}{n} + \frac{2}{\pi n^2} Cn \log n \leq C \frac{\log n}{n}.$$

By taking $n_k = k^2$ we see that $|I_{n_k}| \leq 2C \log k / k^2$. Hence

$$\sum_{k=1}^{\infty} |I_{n_k}| \leq 2C \sum_{k=1}^{\infty} \frac{\log k}{k^2} < \infty$$

and the proof is complete. □

Remark Suppose we now fix $x = 1$ and consider the sequence $x_n = \alpha^n$. Then one can show that x_n is uniformly distributed mod 1 for almost all $\alpha > 1$. However, not a single example of such an α is known! (Indeed, it is not even known whether $(3/2)^n$ mod 1 is dense in $[0, 1]$.)

§2.6 References

The standard text on uniform distribution mod 1 is

L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, J. Wiley, New York, 1974.

The book

W. Parry, *Topics in Ergodic Theory*, C.U.P., Cambridge, 1981

relates uniform distribution mod 1 to ergodic theory. More connections between ergodic theory and uniform distribution can be found in

I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai, *Ergodic Theory*, Springer, Berlin, 1982.

§2.7 Exercises

Exercise 2.1
Show that if x_n is uniformly distributed mod 1 then $\{x_n\}$ is dense in $[0, 1)$.

Exercise 2.2
Can we replace 'continuous' by 'Lebesgue-integrable' in Condition (ii) of Theorem 2.1?

Exercise 2.3
Calculate the frequency with which 2^n has r ($r = 1, \ldots, 9$) as the leading digit of its base 10 representation. (This is a particular case of Benford's law. See `http://en.wikipedia.org/wiki/Benford%27s_law`*.)*

(Hint: first show that 2^n *has leading digit* r *if and only if*

$$r\,10^\ell \leq 2^n < (r+1)10^\ell$$

for some $\ell \in \mathbb{Z}^+$*.)*

Exercise 2.4
Prove Proposition 2.3.

Exercise 2.5
Deduce the general case of Weyl's theorem (where at least one non-constant coefficient is irrational) from the special case proved above (where the leading coefficient is irrational).

Exercise 2.6
Let $p(n) = \alpha_k n^k + \alpha_{k-1}n^{k-1} + \cdots + \alpha_1 n + \alpha_0$, $q(n) = \beta_k n^k + \beta_{k-1}n^{k-1} + \cdots + \beta_1 n + \beta_0$. *Show that* $(p(n), q(n)) \in \mathbb{R}^2$ *is uniformly distributed mod 1 if, for some* $1 \leq i \leq k$, α_i, β_i *and* 1 *are rationally independent.*