Please complete the unit surveys for this course (and your other course units).

- Circles and straight lines in $\mathbf{R}^{2}$ have equations of the form

$$
\alpha z \bar{z}+\beta z+\bar{\beta} \bar{z}+\gamma=0, \quad \alpha, \gamma \in \mathbf{R}, \beta \in \mathbf{C}
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$$

- Vertical straight lines and circles with real centres have equations of the form

$$
\alpha z \bar{z}+\beta z+\beta \bar{z}+\gamma=0, \quad \alpha, \beta, \gamma \in \mathbf{R}
$$

Straight lines in $\mathbf{R}^{2}$ have the equation $a x+b y+c=0, a, b, c \in \mathbf{R}$.

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Substitute in:

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$$
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& a\left(\frac{z+\bar{z}}{2}\right)+b\left(\frac{z-\bar{z}}{2 i}\right)+c=0 \\
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$$

$\beta$ is real iff $b=0$ iff $a x+c=0$ iff $x$ is constant iff the line is vertical.

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- the angle of parallelism formula.
- Angle of parallelism:

Let $\Delta$ be a hyperbolic right-angled triangle with one ideal vertex, internal angles $0, \pi / 2, \alpha$. Suppose the side of finite hyperbolic length has hyperbolic length $a$. Then

$$
\cosh a=\frac{1}{\sin \alpha}
$$

Pythagoras The
Let $\Delta$ be a hyp. ight-angled triangle with sides ot hype length $a, b, c \quad(c=$ "hypotenuse)


$$
\cosh c=\cosh a \cosh b .
$$

Gauvs-Bonnet
Let $D$ be a hyp triangle with internal angles $\alpha, \beta, \gamma$. Then

$$
\operatorname{Ara}_{H+1}(\Delta)=\pi-(\alpha+\beta+\gamma) \text {. }
$$

Clugle a parallelism
Let $\Delta$ he a hyp, riangic with are ideal vertex and with the ore side of finite length having hypertali length $a$. Let $\alpha=$ other angle of $D$.


$$
\cosh a=\frac{1}{\sin \alpha}
$$

PL Andy a Mobile tx so that the picture looms like: (duenret change angles at möbiuntres


- Classifying Möbius transformations:
- Either: calculate the fixed points
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MANCHESTER
The University of Manchester

Complete your unit surveys to tell us about the quality of the teaching you receive

## PLEASE COMPLETE THE UNIT SURVEYS!!!

- Let $\Gamma$ be a Fuchsian group.
- An open set $F \subset \mathbf{H}$ is a fundamental domain for $\Gamma$ if
- $\bigcup_{\gamma \in \Gamma} \gamma(\mathrm{cl} F)=\mathbf{H}$
- $\gamma_{1}(F) \cap \gamma_{2}(F)=\varnothing, \gamma_{1} \neq \gamma_{2}, \gamma_{1}, \gamma_{2} \in \Gamma$
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It's NOT $\bigcup \gamma(\mathrm{cl} F)=\mathbf{H}$. $\gamma \in \Gamma \backslash\{\mathrm{id}\}$

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4. Find $H_{p}(\gamma)$, the half-plane determined by $L_{p}(\gamma)$ that contains $p$.
5. $D(p)=\bigcap_{\gamma \in \Gamma \backslash\{\text { id }\}} H_{p}(\gamma)$ is a fundamental domain.

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- The perpendicular bisector of $\left[z_{1}, z_{2}\right]$ is $\left\{z \in \mathbf{H} \mid d_{\mathbf{H}}\left(z, z_{1}\right)=d_{\mathbf{H}}\left(z, z_{2}\right)\right\}$ and then use the formula for $\cosh d_{\mathbf{H}}(z, w)$.

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- Eye-ball it (and use the geometry of the hyperbolic plane - in particular, reflections in geodesics are isometries (albeit not Möbius transformations).
$1^{\prime}$ bisector of $[i, 4]$


$$
\begin{aligned}
& \text { Recall } \cosh d_{H}(z, \omega)=1+\frac{|z-\omega|^{2}}{2 \ln z \ln \omega} \\
& L^{r} \text { bisected of }\left[i, x_{i}\right]=\left\{z_{E}|H| d_{H}(z, i)=d_{H}\left(z, x_{i}\right)\right\} \\
& \cosh d_{H}\left(z_{i}\right)=\cosh d_{H}\left(z_{1}, y_{i}\right) \\
& \Leftrightarrow 1+\frac{|z-i|^{2}}{\left.2 \operatorname{lm}\right|_{m i}}=1+\frac{|z-4 i|^{2}}{2 \operatorname{lm} z \operatorname{lm} q_{i}} . \\
& \Longleftrightarrow 4|z-i|^{2}=|z-4 i|^{2} \\
& \Leftrightarrow 4(z-i)(\bar{z}+i)=(z-4 i)(\bar{z}+4 i) \\
& \Leftrightarrow 4 z \bar{z}+4 i z-4 i \bar{z}+4=\bar{z} \bar{z}+4 i z-4 i \bar{z}+16 \\
& \Leftrightarrow z \bar{z}=4 \text {, ie }|z|=2 \text {. }
\end{aligned}
$$

$\Sigma_{g}: \perp^{r}$ bisector of $\left[\frac{1}{2}, \frac{1}{2} e^{2 \pi / 3}\right]$ in $\mathbb{D}$.

symmetric abut this gecoleric ( $=$ diameter at angle $\pi / 3$ ).
$\perp^{r}$ bisector = diameter at angle $\pi / 3$.


## Are the following TRUE or FALSE?

1. The set $\{z \in \mathbf{H}||z| \geq 1\}$ is a fundamental domain for some Fuchsian group.
2. There exists a Fuchsian group (acting on $\mathbf{D}$ ) with fundamental domain given by

$$
\left\{z \in \mathbf{D} \left\lvert\,-\frac{\pi}{3}<\arg (z)<\frac{\pi}{3}\right.\right\} .
$$

3. The modular group $\operatorname{PSl}(2, \mathbf{Z})$ has a fundamental domain with hyperbolic area $2 \pi$.
4. Fundamental domains for (non-trivial) Fuchsian groups are unique.
5. The set $\{z \in \mathbf{H}||z| \geq 1\}$ is a fundamental domain for some Fuchsian group.
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## FALSE!!

This is not an open set.
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$$

## TRUE!!!

Take $\Gamma=\left\{\mathrm{id}, \gamma_{1}, \gamma_{2}\right\}$ where $\gamma_{1}=$ rotate through 120 degrees anticlockwise, $\gamma_{2}=$ rotate through 120 degrees clockwise.
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## FALSE!!!

All fundamental domains for a given Fuchsian group have the same area.
We know that $\operatorname{PSl}(2, \mathbf{Z})$ has a hyperbolic triangle as a fundamental domain. By the Gauss-Bonnet Theorem, the hyperbolic area of a triangle must be $\leq \pi$.
[Indeed, the the triangle with vertices at $\infty, \frac{ \pm 1+\sqrt{3}}{2}$ is a fundamental domain for $\operatorname{PSl}(2, \mathbf{Z})$. This has internal angles $0, \pi / 3, \pi / 3$ and so has hyperbolic area $\pi-(\pi / 3+\pi / 3)=\pi / 3$.]
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FALSE!!!

Q1. Find (in the form $\alpha z \bar{z}+\beta z+\beta \bar{z}+\gamma=0, \alpha, \beta, \gamma \in \mathbf{R}$ ) the equation of the straight line through the points $1+3 \mathrm{i}, 1+8 \mathrm{i}$.

Q2(i). Write down a Möbius transformation that maps the geodesic given by $\operatorname{Re}(z)=3$ to the imaginary axis. Find a Möbius transformation that maps the geodesic given by $\operatorname{Re}(z)=3$ to the imaginary axis and maps the point $3+2 i$ to the point $i$.

Q2(ii). Write down a Möbius transformation of $\mathbf{H}$ that maps the geodesic with end-points at $-2,2$ to the imaginary axis.

Q3. What does the following picture, drawn in the Poincaré disc $\mathbf{D}$, look like when drawn in the upper half-plane $\mathbf{H}$ ?

Q4. Suppose two geodesics intersect as illustrated below. Show that $\sin \theta=\frac{2 a b}{a^{2}+b^{2}}$. [Hint: suppose the semi-circle has centre $x$ and radius $r$. Use the (Euclidean) Pythagoras' theorem to relate $r, x, a, b$.]

## Q5. [Adapted from B7, 2017/18.]

(i) Let $\gamma(z)=(a z+b) /(c z+d), a, b, c, d \in \mathrm{R}, a d-b c>0$ be a Möbius transformation of H. Recall that $\gamma$ is parabolic if it has one fixed point in $\partial \mathbf{H}$ and no fixed points in $\mathbf{H}$.

Assume that $c \neq 0$. By considering the equation $\gamma\left(z_{0}\right)=z_{0}$ show that $\gamma$ has a unique fixed point on the boundary (and so is parabolic) If and only if $(d-a)^{2}+4 b c=0$.
(ii) Let $k>0, \ell>1$. Define

$$
\gamma_{1}(z)=\frac{k z}{(k+1) z+1}, \quad \gamma_{2}(z)=\frac{\left(1-\ell^{2}\right) z-2 \ell^{2}}{2 z+\left(1-\ell^{2}\right)}
$$

Consider the quadrilateral with vertices at $0,-1,1$, il as illustrated below. Then $\gamma_{1}, \gamma_{2}$ pair the sides of this quadrilateral as illustrated (you do not need to check this).


Show that there is one elliptic cycle $\mathscr{E}$ and two parabolic cycles $\mathscr{P}_{1}, \mathscr{P}_{2}$.
Determine conditions on $k, \ell, \theta$ that ensure that the elliptic cycle $\mathscr{E}$ satisfies the Elliptic Cycle Condition.

Determine conditions on $k, \ell, \theta$ that ensures that the parabolic cycles $\mathscr{P}_{1}, \mathscr{P}_{2}$ satisfy the Parabolic Cycle Condition.
(iii) In the cases where the ECC and the PCC hold, Poincaré's Theorem tells us that $\gamma_{1}, \gamma_{2}$ generate a Fuchsian group $\Gamma$. Write down a presentation of $\Gamma$ in terms of generators and relations.
(iv) Describe the quotient surface $\mathbf{H} / \Gamma$ (i.e. how many marked points does it have, how many cusps does it have, what is its genus?). Sketch a picture of $\mathbf{H} / \Gamma$.

Q1 $1+3 i, 1+8 i$ lie on a vertical straight line, so we can take $\alpha=0$. We want to find $\beta, \gamma \in \mathbb{R}$ such that $\beta z+\beta \bar{z}+\gamma=0$ holds for $z=1+3 i, 1+8 i$. Substituting in, we get the simultaneous equations

$$
\beta(1+3 i)+\beta(1-3 i)+\gamma=0, \quad \beta(1+8 i)+\beta(1-8 i)+\gamma=0,
$$

i.e. $2 \beta+\gamma=0$. Take $\beta=1, \gamma=-2$ as a solution. Hence $1+3 i, 1+8 i$ lie on the vertical straight line with equation $z+\bar{z}-2=0$.
[Note: we will not get unique solutions for $\beta, \gamma$. This is because any (non-zero) multiple of the equation $\beta z+\beta \bar{z}+\gamma=0$ describes the same vertical straight line.]

Q2 (i) The translation $\gamma_{1}(z)=z-3$ moves the straight line $\operatorname{Re}(z)=3$ to $\operatorname{Re}(z)=0$, the imaginary axis.
Note that $\gamma_{1}(3+2 i)=2 i$. Let $\gamma_{2}(z)=z / 2$. Then $\gamma_{2}$ maps the imaginary axis to itself and $\gamma_{2}(2 i)=i$. Hence $\gamma_{2} \gamma_{1}(z)=(z-3) / 2$ maps the geodesic given by $\operatorname{Re}(z)=3$ to the imaginary axis and the point $3+2 i$ to $i$.
(ii) Take $\gamma(z)=(z-2) /(z+2)$. This is a Möbius transformation (you should check that ' $a d-b c>0$ '). We have $\gamma(2)=0, \gamma(-2)=\infty$. Hence $\gamma$ maps the geodesic with endpoints at $2,-2$ to the geodesic with endpoints at $0, \infty$; the geodesic with endpoints at $0, \infty$ is the imaginary axis.

Q3


Q4 [This is Exercise 5.7 in the notes.]
Suppose that the semi-circular geodesic has radius $r$ and centre $x$ as illustrated. Construct the (Euclidean) right-angled triangle $0, x, i b$.



As the radius of the semi-circle is $r$ we have $|x-i b|=r$ and $|x-a|=r$; hence the base of the right-angled triangle has length $r-a$. By Pythagoras' Theorem we have $(r-a)^{2}+b^{2}=r^{2}$, which expands out and simplifies to $r=\left(a^{2}+b^{2}\right) / 2 a$. From the figure above we have that $\sin \theta=b / r=2 a b /\left(a^{2}+b^{2}\right)$.

Q5 (i) Let $\gamma(z)=(a z+b) /(c z+d)$. We have

$$
\gamma\left(z_{0}\right)=z_{0} \Leftrightarrow \frac{a z_{0}+b}{c z_{0}+d}=z_{0} \Leftrightarrow a z_{0}+b=c z_{0}^{2}+d z_{0} \Leftrightarrow c z_{0}^{2}+(d-a) z_{0}-b=0 .
$$

This is a quadratic (as $c \neq 0)$. The discriminant is $(d-a)^{2}+4 b c$ and this quadratic has a unique real solution (and so $\gamma$ is parabolic) precisely when $(d-a)^{2}+4 b c=0$. [Note: the reason for including this question is because using it will make your life easier later on...]
(ii)

$$
\binom{i \ell}{s_{4}} \xrightarrow{\gamma_{2}}\binom{i \ell}{s_{3}} \xrightarrow{*}\binom{i \ell}{s_{4}}
$$

Hence we have an elliptic cycle $\mathcal{E}: i \ell$, with corresponding elliptic cycle transformation $\gamma_{2}$ and angle $\operatorname{sum} \operatorname{sum}(\mathcal{E})=2 \theta$.
The elliptic cycle condition will hold if there exists an integer $m \geq 1$ such that $m \operatorname{sum}(\mathcal{E})=2 \pi$, i.e. if $2 m \theta=2 \pi$, i.e. if $\theta=\pi / m$ for some integer $m \geq 1$. If $m=1$ then $\theta=\pi$; the angle in the picture would then be $2 \pi$ which is impossible. Hence $m \geq 2$.

$$
\binom{0}{s_{1}} \xrightarrow{\gamma_{1}}\binom{0}{s_{2}} \xrightarrow{*}\binom{0}{s_{1}}
$$

Hence we have a parabolic cycle $\mathcal{P}_{1}: 0$ with corresponding parabolic cycle transformation $\gamma_{1}$.
The parabolic cycle condition will hold for $\mathcal{P}_{1}$ if $\gamma_{1}$ is either parabolic or the identity.
We can use part (i) of the question to test if $\gamma_{1}$ is parabolic. Here $a=k, b=$ $0, c=k+1, d=1$. We want to have

$$
(d-a)^{2}+4 b c=(1-k)^{2}=0
$$

and this holds if and only if $k=1$. Hence the PCC holds if and only if $k=1$. [Alternatively, we could have looked for fixed points of $\gamma_{1}$. We have

$$
\gamma_{1}\left(z_{0}\right)=z_{0} \Leftrightarrow \frac{k z_{0}}{(k+1) z_{0}+1}=z_{0} \Leftrightarrow(k+1) z_{0}^{2}+(1-k) z_{0}=0 \Leftrightarrow z_{0}=0, \frac{1-k}{1+k}
$$

and this has a unique fixed point on $\partial \mathbb{H}$ iff $k=1$.]
[Note that $\gamma_{1}$ as defined in the question is not in normalised form so, had you chosen to calculate $\tau\left(\gamma_{1}\right)$ then you would have had to have normalised it first; the algebra here gets a bit messy.]

$$
\binom{-1}{s_{4}} \xrightarrow{\gamma_{1}}\binom{1}{s_{3}} \xrightarrow{*}\binom{1}{s_{2}} \xrightarrow{\gamma_{2}^{-1}}\binom{-1}{s_{1}} \xrightarrow{*}\binom{-1}{s_{4}}
$$

Hence we have a parabolic cycle $\mathcal{P}_{2}:-1 \rightarrow 1$ with corresponding parabolic cycle transformation $\gamma_{2}^{-1} \gamma_{1}$.
The parabolic cycle condition will hold for $\mathcal{P}_{2}$ if $\gamma_{2}^{-1} \gamma_{1}$ is either parabolic or the identity.
We can use part (i) of the question to test if $\gamma_{2}^{-1} \gamma_{1}$ is parabolic. [Alternatively, we could calculate explicitly the fixed points of $\gamma_{2}^{-1} \gamma_{1}$, or calculate $\tau\left(\gamma_{2}^{-1} \gamma_{1}\right)$.] Note that we need the PCC to hold for $\mathcal{P}_{1}$, so there is no loss in generality in assuming that $k=1$. By using the connection between matrices and Möbius transformations, $\gamma_{2}^{-1} \gamma_{1}$ has matrix

$$
\left[\begin{array}{cc}
1-\ell^{2} & 2 \ell^{2} \\
-2 & 1-\ell^{2}
\end{array}\right]\left[\begin{array}{ll}
1 & 0 \\
2 & 1
\end{array}\right]=\left[\begin{array}{cc}
1+3 \ell^{2} & 2 \ell^{2} \\
-2 \ell^{2} & 1-\ell^{2}
\end{array}\right]
$$

This is not normalised, and the easiest way to check if this is parabolic is to use part (i) of this question. Hence for $\gamma_{2}^{-1} \gamma_{1}$ to be parabolic, and so for the PCC to hold for $\mathcal{P}_{2}$, we want

$$
(d-a)^{2}+4 b c=\left[\left(1-\ell^{2}\right)-\left(1-3 \ell^{2}\right)\right]^{2}+4\left(2 \ell^{2}\right)\left(-2 \ell^{2}\right)=16 \ell^{4}-16 \ell^{4} \equiv 0
$$

Hence the PCC always holds, irrespective of the value of $\ell$.
(iii) There are two generators $a, b$ (corresponding to $\gamma_{1}, \gamma_{2}$, respectively). There is one relation corresponding to the elliptic cycle $\mathcal{E}$. Suppose $\theta$ is such that $\theta=\pi / \mathrm{m}$ for $m \geq 2$ then we have presentation $\Gamma=\left\langle a, b \mid a^{m}=e\right\rangle$.
(iv) Recall that we must have that $m \geq 2$. Hence the elliptic cycle $\mathcal{E}$ is non-accidental and there is one marked point of order $m$.
There are two parabolic cycles, hence $\mathbb{H} / \Gamma$ has two cusps.
To calculate the genus, one can either think geometrically or think in terms of Euler's formula. Geometrically, the quadrilateral and its side-pairings are similar to taking a (Euclidean) square and (thinking of $s_{1}, s_{2}, s_{3}, s_{4}$ as the bottom, right, top and left edges respectively) gluing the bottom edge to the right edge and the left edge to the top edge. This makes something that, topologically, looks like a sphere and-as it has no holes in it-has genus $g=0$.
Alternatively one can use Euler's formula as follows. The quadrilateral forms a triangulation of $\mathbb{H} / \Gamma$ with $V=3$ vertices (one elliptic and two parabolic cycles), $E=2$ edges (4 paired sides) and $F=1$ face. Hence $2-2 g=V-E+F=$ $3-2+1=2$, so $g=0$.
Hence $\mathbb{H} / \Gamma$ has two fixed points, 1 marked point of order $m$, and genus 0 . It looks like the surface of a 'sausage', albeit with both ends stretched out to form cusps, together with one kink in it, corresponding to the marked point.

