What did we do last time?

Stated Poincaré's Thm (case of boundary vertices, but no free edges)

Suppose:
- D is a convex hyp polygon equipped with a set of side-pairing txs
- no side is paired with itself
- all elliptic cycles satisfy the ECC
- all parabolic cycles satisfy the PCC

Then:
- the side-pairing txs generate a Fuchsian group Γ
- D is a fundamental domain for Γ
- one can give a presentation for Γ in terms of generators & relations.

What will we do today?

Define the signature of a Fuchsian group Γ & the quotient space $\mathbb{H}/\Gamma$.

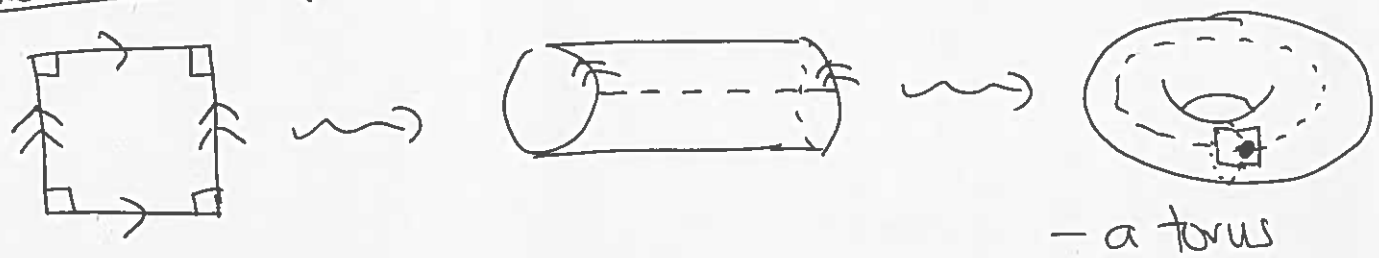
Relate the geometry of $\mathbb{H}/\Gamma$ with the algebra of Γ.

21. The signature of a fuchsian group

Let Γ be a fuchsian group. Suppose D is a Dirichlet polygon with $\text{Area}_{\mathbb{H}}(D) < \infty$. Equip D with a set of side-pairing txs.

<u>Idea</u>: Form a surface $\mathbb{H}/\Gamma$ by gluing together paired sides

<u>Euclidean example</u>



– a torus

Note: the corners glue together to give ~~a~~ one point on the torus with total angle 2π.

<u>How to construct $\mathbb{H}/\Gamma$</u>

Let $\mathcal{E}$ be an elliptic cycle. All the vertices on $\mathcal{E}$ glue together to give a single point on $\mathbb{H}/\Gamma$. The total angle at this point is $\text{sum}(\mathcal{E})$.

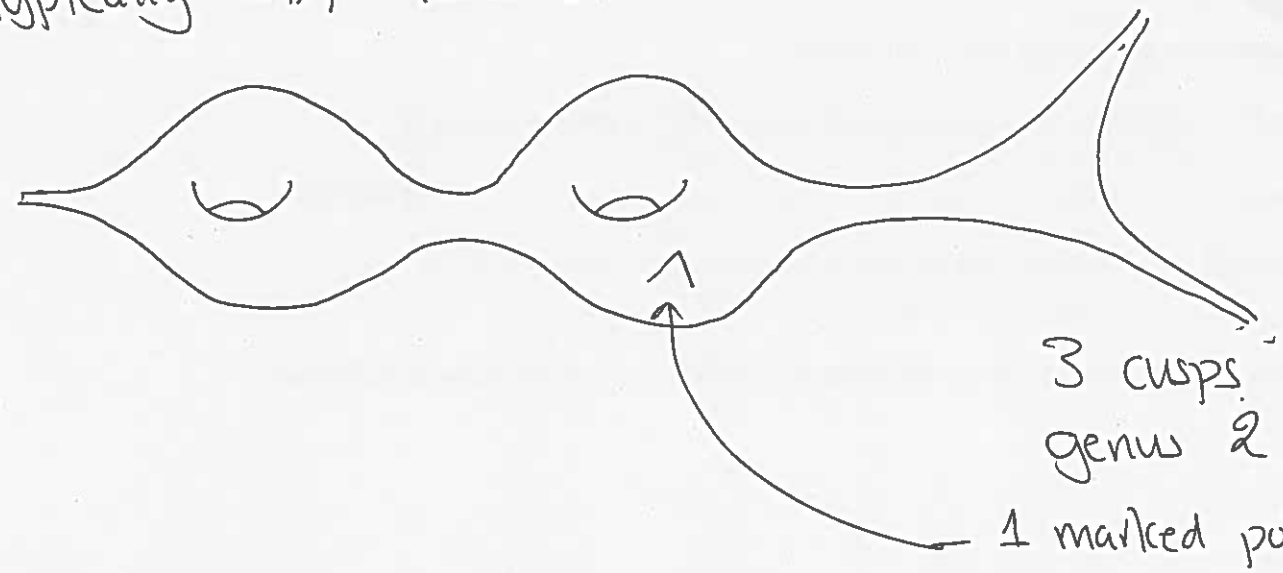
Recall: $\text{sum}(\mathcal{E}) = 2\pi \iff$ order of $\mathcal{E}$ is 1. $\iff$ $\mathcal{E}$ is an <u>accidental cycle</u>
$m(\mathcal{E}) = 1$

If $\text{sum}(\mathcal{E}) < 2\pi$ (equivalently $\mathcal{E}$ has order $m(\mathcal{E}) > 1$) then we have a point on $\mathbb{H}/\Gamma$ with total angle $\text{sum}(\mathcal{E}) = \cancel{2\pi/m(\mathcal{E})}$ $2\pi/m$, where $m = m(\mathcal{E})$. We call this a <u>marked point of order m</u>.

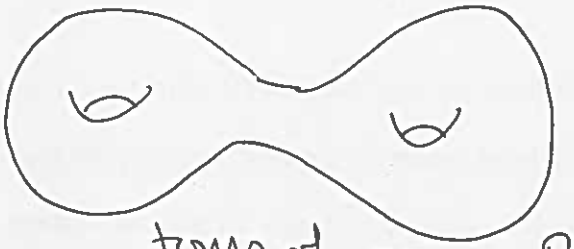
Marked points look like "kinks" in the surface $\mathbb{H}/\Gamma$.

Let $\mathcal{P}$ be a parabolic cycle. Each vertex on $\mathcal{P}$ glues together & the parabolic cycle gives a <u>cusp</u> on $\mathbb{H}/\Gamma$.

Typically H/Γ looks like:



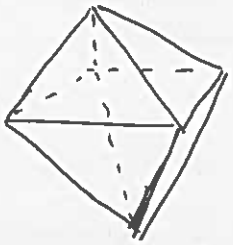
<u>Example</u> Let D be a regular hyp. octagon with internal angle $\pi/4$. This has one elliptic cycle $\mathcal{E}$ and $m(\mathcal{E}) = 1$. ($\mathrm{sum}(\mathcal{E}) = 2\pi$).



torus of genus 2.
(no marked points on the elliptic cycle $\mathcal{E}$ has $\mathrm{sum}(\mathcal{E}) = 2\pi$ – ie $\mathcal{E}$ is accidental.)

<u>Euler characteristic</u>

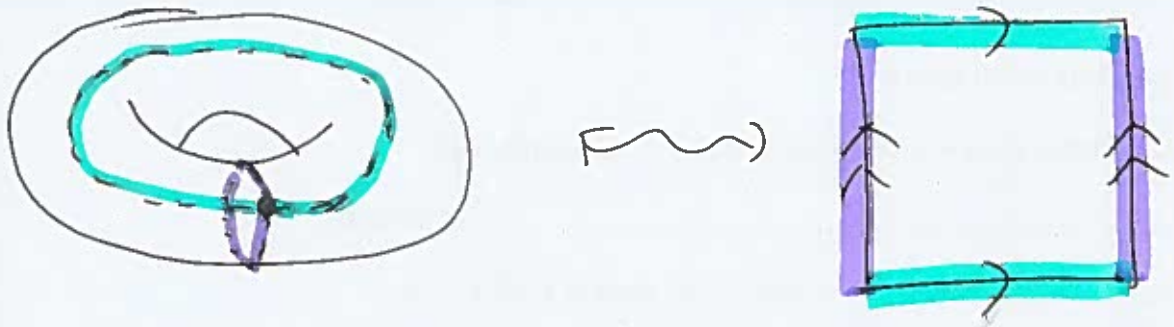
Let X be a 2-dimensional space. Triangulate X into finitely many polygons with V vertices, $\mathcal{E}$ edges, F faces (= number of polygons).



$V = 6 \quad \mathcal{E} = 12. \quad F = 8$

The Euler characteristic of X is $\chi(X) = V - \mathcal{E} + F$.

In the above example $\chi(X) = 6 - 12 + 8 = 2$.



$V = 1. \qquad \mathcal{E} = 2. \qquad F = 1$

$\chi(\text{torus of genus } 1) = V - \mathcal{E} + F = 1 - 2 + 1 = 0.$

<u>Defn</u> The <u>genus</u> g of X is $\chi(X) = 2 - 2g$.
Topologically, the genus is the number of "holes" in X

<u>Defn</u> Suppose Γ is a Fuchsian group with Dirichlet poly. D & suppose D has no vertices on the boundary. Then we call Γ <u>cocompact</u>.

<u>Defn</u> Let Γ be a cocompact Fuchsian group.
Let g = genus of $\mathbb{H}/\Gamma$.
Suppose there are k elliptic cycles $\mathcal{E}_1, \ldots, \mathcal{E}_k$.
Suppose $\mathcal{E}_j$ has order m_j $\quad (m_j \times \text{sum}(\mathcal{E}_j) = 2\pi)$.

Suppose $\mathcal{E}_1, \ldots, \mathcal{E}_r$ are non-accidental $\quad (m_j > 1)$
$\mathcal{E}_{r+1}, \ldots, \mathcal{E}_k$ are accidental $\quad (m_j = 1)$.

The <u>signature of Γ</u> is $\text{sig}(\Gamma) = (g; m_1, \ldots, m_r)$
$= (\text{genus of } \mathbb{H}/\Gamma;$ orders of the <u>non-accidental</u> elliptic cycles$)$.

(If there are no non-accidental elliptic cycles then we write $\text{sig}(\Gamma) = (g; -)$.)

The signature $sig(\Gamma)$ ~~tells~~ tells us a lot about the geometry of D, and $\mathbb{H}/\Gamma$, and the algebra of Γ.

Example:

<u>Proposition</u> Let Γ be a cocompact Fuchsian group with $sig(\Gamma) = (g; m_1, \quad, m_r)$. Let D be any fundamental domain for Γ. Then

$$\text{Area}_{\mathbb{H}}(D) = 2\pi\left[(2g-2) + \sum_{j=1}^{r}\left(1 - \frac{1}{m_j}\right)\right]$$

<u>Corollary</u> Let Γ be a cocompact Fuchsian group. Let D be a fundamental domain for Γ. Then

$$\text{Area}_{\mathbb{H}}(D) \geq \frac{\pi}{21}, \text{ with equality iff } sig(\Gamma) = (0; 2, 3, 7).$$

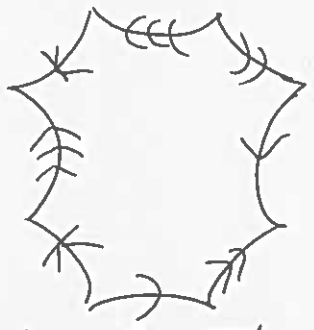
What did we do last time?

$$\operatorname{sig}\Gamma = (g\,;\, \underbrace{m_1, \ldots, m_r})$$

genus (= # of holes) of $\mathbb{H}/\Gamma$

orders of NON-ACCIDENTAL elliptic cycles.

Eg:



all angles $\pi/4$

$\rightsquigarrow$

torus of genus 2.

- one accidental cycle.

Euler characteristic $\chi = V - E + F = 2 - 2g$

What will we do today?

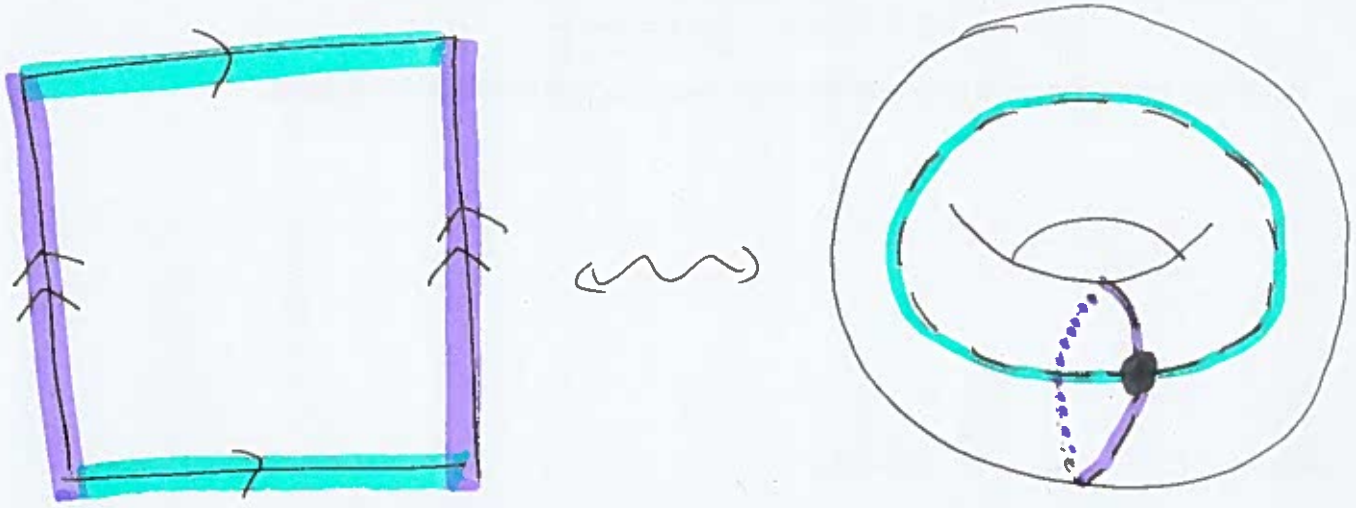
- Let Γ be a cocompact Fuchsian group.

  $\operatorname{sig}(\Gamma) = (g\,;\, m_1, \ldots, m_r)$

  D = any fundamental domain for Γ

  Then $\operatorname{Area}_{\mathbb{H}}(D) = 2\pi \left[(2g-2) + \sum_{j=1}^{r} \left(1 - \frac{1}{m_j}\right) \right]$

- Sketch how to construct a Fuchsian group with given signature.



$$V = 1$$
$$E = 2$$
$$F = 1$$

$$V - E + F = 1 - 2 + 1 = 0$$
$$= 2 - 2g \qquad g = 1.$$

Pf that $\text{Area}_{\mathbb{H}}(D) = 2\pi\left[(2g-2) + \sum_{j=1}\left(1-\frac{1}{m_j}\right)\right]$. ①

All fundamental domains for a given fuchsian group Γ have the same area, so it's sufficient to prove this for a Dirichlet polygon. Let $D = D(p)$ be a Dirichlet polygon with n sides (no side is paired with itself).

Let $\mathcal{E}_1, \ldots, \mathcal{E}_r$ be the non-accidental elliptic cycles. Suppose the order of $\mathcal{E}_j$ is m_j, so $m_j \times \text{sum}(\mathcal{E}_j) = 2\pi$.

Suppose there are s accidental cycles. If $\mathcal{E}$ is accidental then $\text{sum}(\mathcal{E}) = 2\pi$.

Recall:
- vertices in D on a given elliptic cycle glue together to give one point in $\mathbb{H}/\Gamma$. This point will be a vertex in our triangulation of $\mathbb{H}/\Gamma$.
- paired sides in D glue together to give one edge in $\mathbb{H}/\Gamma$.

Also recall: Gauss-Bonnet: $\text{Area}_{\mathbb{H}}(D) = (n-2)\pi - \sum \text{internal angles}$.

$$= (n-2)\pi - \sum_{\substack{\text{elliptic} \\ \text{cycles } \mathcal{E}}} \text{sum}(\mathcal{E}).$$

$$= (n-2)\pi - \left(\sum_{j=1}^{r} \frac{2\pi}{m_j} + 2\pi s\right) \quad ①.$$

D triangulates $\mathbb{H}/\Gamma$ with $V = r+s$ vertices, $\mathcal{E} = n/2$ edges, $F = 1$ faces.

By Euler's formula: $\chi(\mathbb{H}/\Gamma) = 2-2g = V - \mathcal{E} + f$

$$= r + s - \frac{n}{2} + 1 \quad ②.$$

Substitute for s from ② into ① to get the result $\square$.

<u>Prop</u> Let Γ be a cocompact fuchsian group with $\mathrm{sig}(\Gamma) = (g; m_1, \ldots, m_r)$. Let D be a fundamental domain for Γ. Then $\mathrm{Area}_{\mathbb{H}}(D) \geq \frac{\pi}{21}$ with equality iff $\mathrm{sig}(\Gamma) = (0; 2,3,7)$.

<u>Pf (Sketch)</u> It's sufficient to prove

$$2g - 2 + \sum_{j=1}^{r} \left(1 - \frac{1}{m_j}\right) \geq \frac{1}{42}. \qquad (*)$$

<u>If $g > 1$</u>: Then $2g - 2 > 1 > \frac{1}{42}$, so $(*)$ holds.

<u>If $g = 1$</u>: Then $2g - 2 = 0$. Note $m_j \geq 2$. So LHS of $(*)$

$$\geq \sum_{j=1}^{r} \left(1 - \frac{1}{m_j}\right) \geq 1 - \frac{1}{2} \geq \frac{1}{42}, \text{ so } (*) \text{ holds.}$$

<u>If $g = 0$</u>: Then $2g - 2 = -2$.

<u>If $r \geq 5$</u>. LHS of $(*) \geq -2 + \sum_{j=1}^{5} \underbrace{\left(1 - \frac{1}{m_j}\right)}_{\geq 1/2} \geq -2 + \frac{5}{2} = \frac{1}{2} \geq \frac{1}{42}$ so $(*)$ holds.

<u>If $r = 4$</u>: The minimum of LHS of $(*)$ occurs when $\mathrm{sig}(\Gamma) = (0; 2, 2, 2, 3)$ & the LHS is still $\geq 1/42$.

<u>If $r = 3$</u> Let $\mathrm{sig}(\Gamma) = (0; k, l, m)$. Then $(*)$ holds

$$\iff 1 - \left(\frac{1}{k} + \frac{1}{l} + \frac{1}{m}\right) \geq \frac{1}{42}.$$

& equality holds iff $(k, l, m) = (2, 3, 7)$.

(this is a really dull calculation!) $\square$

Note: If Γ is cocompact & $\mathrm{sig}(\Gamma) = (g; m_1, \ldots, m_r)$

then $$2g-2+\sum_{j=1}^{r}\left(1-\frac{1}{m_j}\right) > 0.$$

Conversely we have

<u>Thm</u> If $(g; m_1, \ldots, m_r)$ is s.t. $2g-2+\sum_{j=1}^{r}\left(1-\frac{1}{m_j}\right) > 0$

then $\exists$ a cocompact Fuchsian group Γ with $\mathrm{sig}(\Gamma) = (g; m_1, \ldots, m_r)$.

<u>Example</u> $\forall g \geq 2$, there exist a cocompact Fuchsian group Γ s.t. $\mathrm{sig}(\Gamma) = (g; -)$. Here $\mathbb{H}/\Gamma$ is a torus of genus g

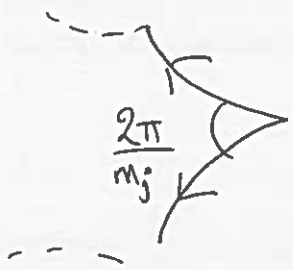
(g holes).

<u>Sketch of the main idea in the proof:</u>

- Want to generate holes / handles

– need $4g$ sides in D to get genus g in $\mathbb{H}/\Gamma$.

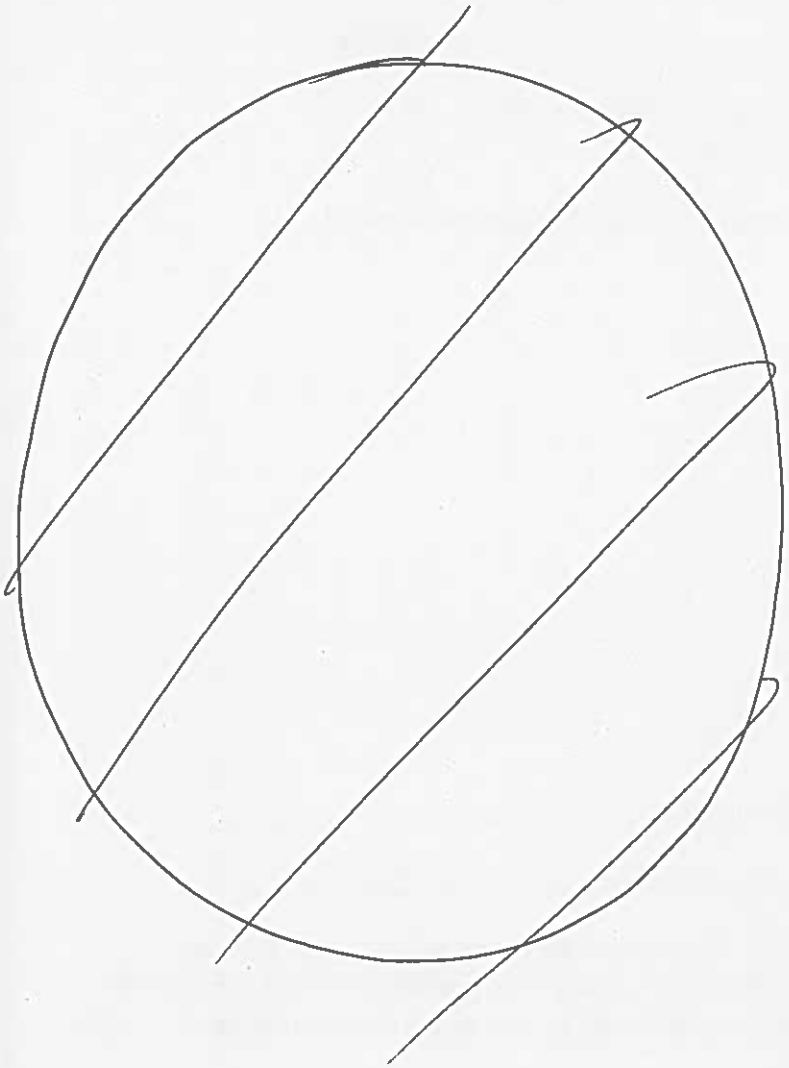
- Want to get marked points of order m_j.

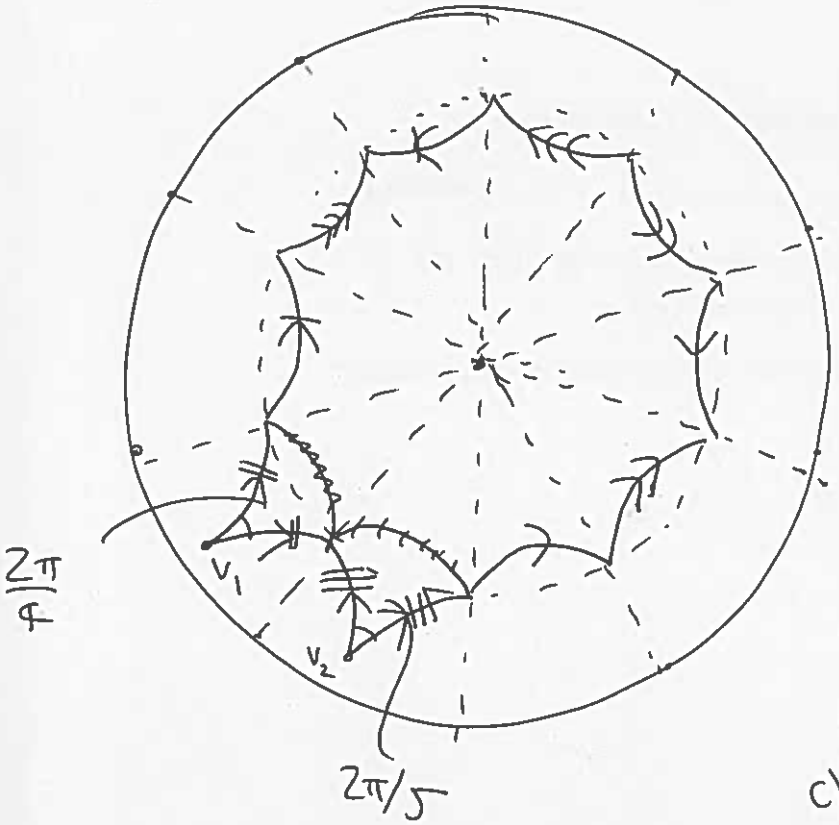


– gives together to give a marked point of order m_j

– need $2r$ sides to get r marked points.

<u>Example</u> Construct a Fuchsian group Γ with $\mathrm{sig}(\Gamma) = (2; 4, 5)$.





Drawing a regular hyperbolic 10-gon.

v_1, v_2 each lie on non-acc. elliptic cycles of orders 4, 5 respectively

The other ~~sides~~ vertices lie on just one elliptic cycle. By choosing radius of the dotted circle carefully, this elliptic cycle is accidental.

Apply Poincaré's Thm to get a fuchsian group Γ with $\mathrm{sig}(\Gamma) = (2; 4, 5)$.

Where could we go next?

"Most surfaces are tori"

Möbius Classification Theorem (1863)

Let S be a compact orientable surface without boundary. Then S is either

- a sphere
- a torus of genus 1
- a torus of genus $g \geq 2$.

Diquet's formula

The curvature $K(x)$ at a point x on a surface is given by $K(x) = \lim_{r \to 0} \frac{12}{\pi} \left[\frac{\pi r^2 - \text{Area } B(x,r)}{r^4} \right]$

(a sphere has curvature > 0, the Euclidean plane $= 0$, hyp. plane -1).

"For surfaces, we only need to think about spheres, the Euclidean plane & the hyperbolic plane."

Poincaré-Koebe Uniformisation Thm (1882, 1907)

Let S be a compact orientable surface without boundary. Suppose S has constant curvature. Then $\exists$ a covering surface M & a discrete group of isometries Γ s.t. $S \cong M/\Gamma$ and:

(1) if S has +ive curvature then $M =$ sphere
(2) if S has 0 curvature then $M = \mathbb{R}^2$
(3) if S has −ive curvature then $M = \mathbb{H}$.

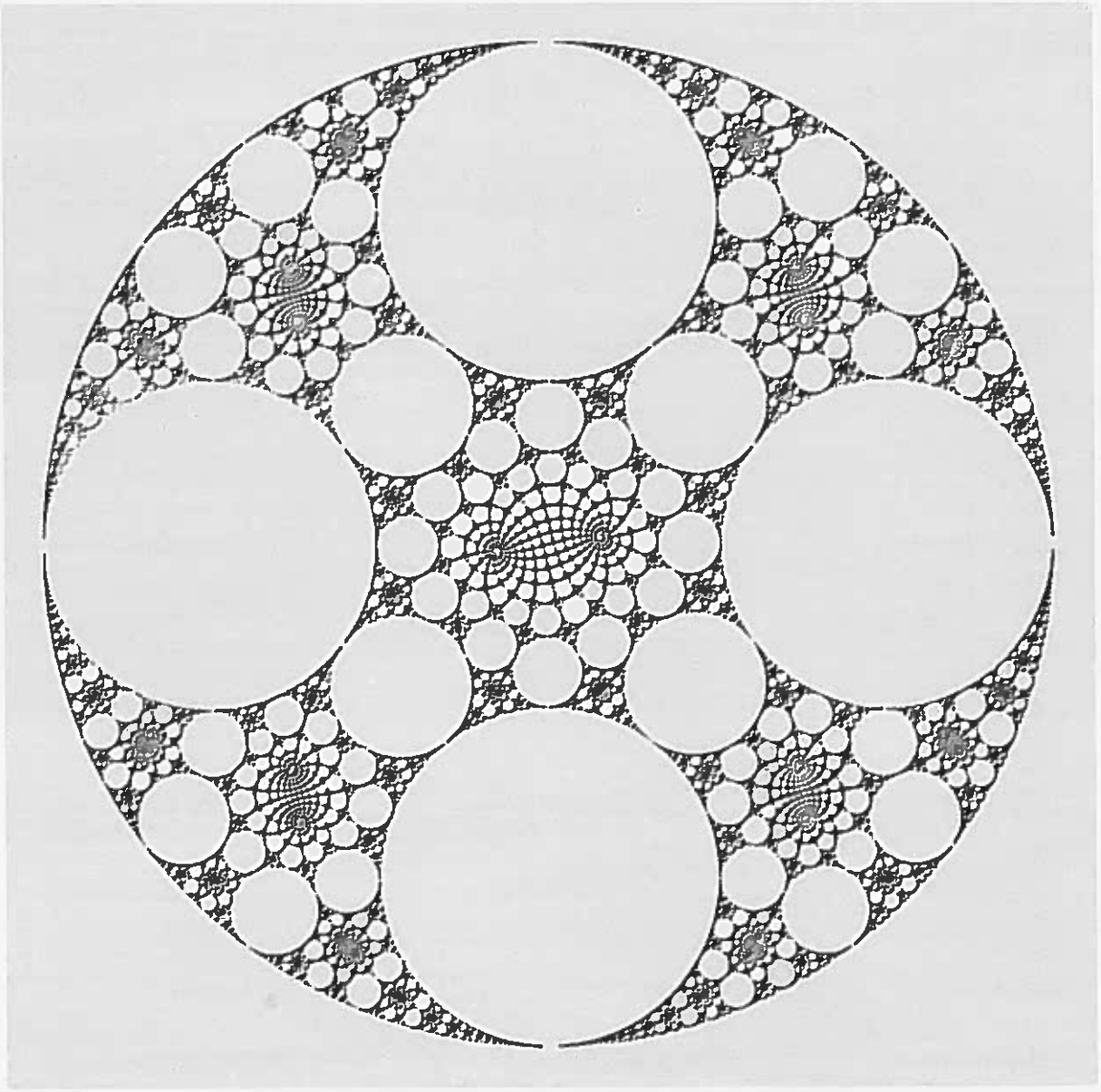
Higher-dimensional hyperbolic geometry

$$\mathbb{H}^n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\} \qquad (\mathbb{H}^2 = \mathbb{H})$$

$$\partial\mathbb{H}^n = \{(x_1, x_2, \ldots, x_{n-1}, 0) \in \mathbb{R}^n\} \cup \{\infty\}$$

Applications to:

- dynamical systems & chaos theory
- fractal geometry



B5.

(i) Recall that the set of Möbius transformations of $\mathbb{D}$ is defined to be

$$\text{Möb}(\mathbb{D}) = \left\{ \gamma : \mathbb{D} \to \mathbb{D} \mid \gamma(z) = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}, \alpha, \beta \in \mathbb{C}, |\alpha|^2 - |\beta|^2 > 0 \right\}.$$

Let $\gamma_1, \gamma_2 \in \text{Möb}(\mathbb{D})$ and write

$$\gamma_1(z) = \frac{\alpha_1 z + \beta_1}{\overline{\beta_1} z + \overline{\alpha_1}}, \qquad \gamma_2(z) = \frac{\alpha_2 z + \beta_2}{\overline{\beta_2} z + \overline{\alpha_2}}.$$

Show that the composition $\gamma_1\gamma_2$ is a Möbius transformation of $\mathbb{D}$.

Show that γ_1^{-1} is a Möbius transformation of $\mathbb{D}$.

[10 marks]

(ii) Recall that if $\sigma : [a, b] \to \mathbb{D}$ is a parametrisation of a path in $\mathbb{D}$ then the hyperbolic length of σ is defined to be

$$\text{length}_{\mathbb{D}}(\sigma) = \int_a^b \frac{2}{1 - |\sigma(t)|^2} |\sigma'(t)| \, dt.$$

How can the hyperbolic lengths of paths then be used to define a metric $d_{\mathbb{D}}$ on $\mathbb{D}$? (You do not need to prove that $d_{\mathbb{D}}$ is a metric.)

[2 marks]

(iii) Let $a \in (0, 1)$ and consider the path σ along the imaginary axis that joins 0 and ia. Write down a parametrisation of σ. Hence show that

$$\text{length}_{\mathbb{D}}(\sigma) = \log\left(\frac{1+a}{1-a}\right).$$

[6 marks]

(iv) Hence show that $d_{\mathbb{D}}(0, ia) = \log\left(\frac{1+a}{1-a}\right)$.

[8 marks]

(v) Find the hyperbolic mid-point of the arc of geodesic in $\mathbb{D}$ between 0 and $4i/5$.

[4 marks]

A4.

(i) Consider the regular hyperbolic decagon in Figure 1 below with each internal angle equal to $\pi/9$ and with the sides paired as illustrated (you may assume that such a hyperbolic decagon exists).

Show that there are two elliptic cycles and determine their orders. By using Poincaré's Theorem, show that the side pairing transformations generate a co-compact Fuchsian group Γ. (You do not need to give a presentation of Γ in terms of generators and relations.)

[10 marks]

(ii) Write down the signature $\mathrm{sig}(\Gamma)$ of Γ. Sketch a picture of the quotient space $\mathbb{H}/\Gamma$.

[4 marks]

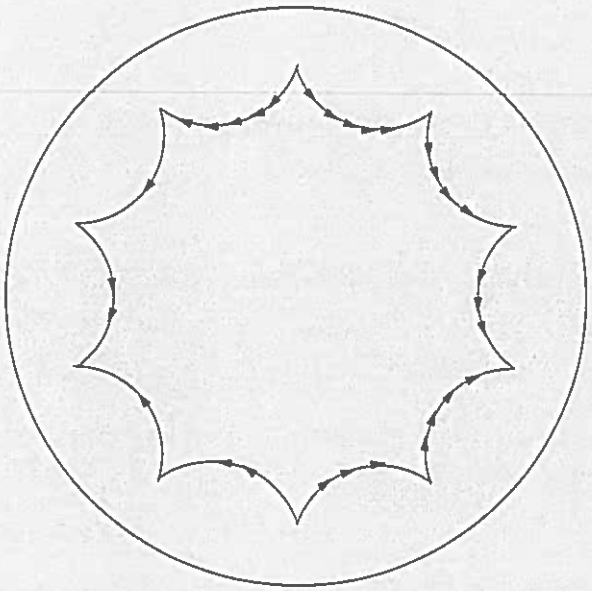


Figure 1: See Question A4. Each internal angle is $\pi/9$ and the sides are paired as indicated.

B5 (i) We have

$$\begin{aligned}
\gamma_1(\gamma_2(z)) &= \frac{\alpha_1\left(\frac{\alpha_2 z+\beta_2}{\bar\beta_2 z+\bar\alpha_2}\right)+\beta_1}{\bar\beta_1\left(\frac{\alpha_2 z+\beta_2}{\bar\beta_2 z+\bar\alpha_2}\right)+\bar\alpha_1} \\
&= \frac{\alpha_1(\alpha_2 z+\beta_2)+\beta_1(\bar\beta_2 z+\bar\alpha_2)}{\bar\beta_1(\alpha_2 z+\beta_2)+\bar\alpha_1(\bar\beta_2 z+\bar\alpha_2)} \\
&= \frac{(\alpha_1\alpha_2+\beta_1\bar\beta_2)z+(\alpha_1\beta_2+\beta_1\bar\alpha_2)}{(\bar\beta_1\alpha_2+\bar\alpha_1\bar\beta_2)z+(\bar\beta_1\beta_2+\bar\alpha_1\bar\alpha_2)} \\
&= \frac{\alpha_3 z+\beta_3}{\bar\beta_3 z+\bar\alpha_3}.
\end{aligned}$$

This is a Möbius transformation of $\mathbb{D}$ as

$$\begin{aligned}
|\alpha_3|^2-|\beta_3|^2 &= (\alpha_1\alpha_2+\beta_1\bar\beta_2)(\bar\beta_1\beta_2+\bar\alpha_1\bar\alpha_2)-(\alpha_1\beta_2+\beta_1\bar\alpha_2)(\bar\beta_1\alpha_2+\bar\alpha_1\bar\beta_2) \\
&= (|\alpha_1|^2-|\beta_1|^2)(|\alpha_2|^2-|\beta_2|^2)>0.
\end{aligned}$$

[You would also have got full marks if you'd exploited the fact that composition of Möbius transformations corresponds to multiplying matrices together, provided that you'd stated this correspondence.]

For the inverse, if $w=\gamma_1(z)$ then

$$w=\frac{\alpha_1 z+\beta_1}{\bar\beta_1 z+\bar\alpha_1} \Leftrightarrow w(\bar\beta_1 z+\bar\alpha_1)=\alpha_1 z+\beta_1 \Leftrightarrow (\bar\beta_1 w-\alpha_1)z=-\bar\alpha_1 w+\beta_1 \Leftrightarrow z=\frac{-\bar\alpha_1 w+\beta_1}{\bar\beta_1 w-\alpha_1}.$$

Hence

$$\gamma_1^{-1}(z)=\frac{-\bar\alpha_1 z+\beta_1}{\bar\beta_1 z-\alpha_1}$$

which is a Möbius transformation of $\mathbb{D}$ as

$$-\bar\alpha_1\times(-\alpha_1)-\beta_1\bar\beta_1=|\alpha_1|^2-|\beta_1|^2>0.$$

[Again, you could also have used matrices to do this.]

(ii) Let $z,w\in\mathbb{D}$. We define

$$d_{\mathbb{D}}(z,w)=\inf\{\text{length}_{\mathbb{D}}(\sigma)\mid \sigma \text{ is a piecewise differentiable path from } z \text{ to } w\}.$$

(iii) A parametrisation of the arc of imaginary axis from 0 to ia is given by $\sigma(t)=it$, $0\le t\le a$.

[There are other formulae that work (eg $\sigma(t)=iat$, $0\le t\le 1$), but this is the simplest and will make life easier in the calculation below.]

We have $\sigma'(t)=i$ and $|\sigma(t)|=t$. Hence

$$\begin{aligned}
\text{length}_{\mathbb{D}}(\sigma) &= \int_0^a \frac{2}{1-t^2}\times 1\,dt \\
&= \int_0^a \frac{1}{1-t}+\frac{1}{1+t}\,dt \\
&= (-\log(1-t)+\log(1+t))|_0^a \\
&= \log(1+a)-\log(1-a) \\
&= \log\frac{1+a}{1-a}.
\end{aligned}$$

[Here we did the integral using partial fractions. If you hadn't realised/remembered that partial fractions was the best way to integrate $1/(1-t^2)$ then you could have gotten a hint from the question. You're given that the answer involves a log. You would get a term $\log(1+t)$ by integrating $1/(1+t)$, and similarly for the $(1-t)$ term. This suggests that you need to look for $1/(1-t)$ and $1/(1+t)$ in the integrand.]

(iv) [This is very similar to Exercise 6.1(iii) in the notes.]
Let $\sigma(t)$ be any path from 0 to ia, $0 \le t \le 1$. Write $\sigma(t) = x(t) + iy(t)$. Then $y(0) = 0, y(1) = a$. Also note that $\sigma'(t) = x'(t)+iy'(t)$ and $|\sigma(t)|^2 = x(t)^2+y(t)^2$. Hence

$$\begin{aligned}
\text{length}_{\mathbb{D}}(\sigma) &= \int_0^1 \frac{2}{1-(x(t)^2+y(t)^2)} \times (x'(t)^2+y'(t)^2)^{1/2}\,dt \\
&\ge \int_0^1 \frac{2y'(t)}{1-y(t)^2}\,dt
\end{aligned}$$

where we have used the facts that $(x'(t)^2+y'(t)^2)^{1/2} \ge y'(t)$ (as $x'(t)^2 \ge 0$) and $1-(x(t)^2+y(t)^2) \le 1-y(t)^2$ (as $x'(t)^2 \ge 0$). Using partial fractions again we have

$$\begin{aligned}
\text{length}_{\mathbb{D}}(\sigma) &= \ge \int_0^1 \frac{y'(t)}{1-y(t)} + \frac{y'(t)}{1+y(t)}\,dt \\
&= (-\log(1-y(t)) + \log(1+y(t)))\big|_0^1 \\
&= \log\frac{1+y(0)}{1-y(0)} \\
&= \log\frac{1+a}{1-a}.
\end{aligned}$$

Combining this with the results in (iii) and the definition given in (ii), we see that $d_{\mathbb{D}}(0, ia) = \log(1+a)/(1-a)$.

(v) The hyperbolic mid-point must lie on the imaginary axis, as this is the geodesic from 0 to $4i/5$. Suppose it occurs at ai. Then $d_{\mathbb{D}}(0, ai) = \frac{1}{2}d_{\mathbb{D}}(0, 4i/5)$. By (iv) we have

$$\log\frac{1+a}{1-a} = \frac{1}{2}\log\frac{1+4/5}{1-4/5} = \frac{1}{2}\log\frac{9/5}{1/5} = \frac{1}{2}\log 9 = \log 9^{1/2} = \log 3.$$

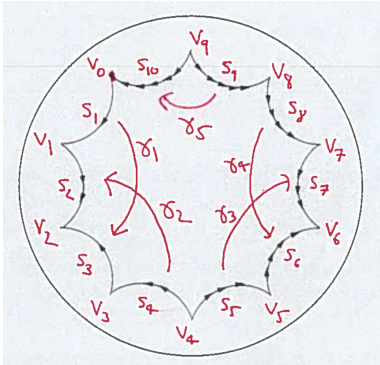
Hence

$$\frac{1+a}{1-a} = 3$$

i.e. $1+a = 3(1-a)$, equivalently $a = 1/2$. Hence the hyperbolic mid-point occurs at $i/2$.

[Note that the numbers were chosen to work out nicely. This is deliberate: I'm assessing you on whether you've learned some hyperbolic geometry, not if you can do arithmetic!]

A4 (i) Label the diagram as shown below.



We have the elliptic cycle

$$
\begin{aligned}
\begin{pmatrix} v_0 \\ s_1 \end{pmatrix} &\xrightarrow{\gamma_1} \begin{pmatrix} v_3 \\ s_3 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} v_3 \\ s_4 \end{pmatrix} \\
&\xrightarrow{\gamma_2} \begin{pmatrix} v_2 \\ s_2 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} v_2 \\ s_3 \end{pmatrix} \\
&\xrightarrow{\gamma_1^{-1}} \begin{pmatrix} v_1 \\ s_1 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} v_1 \\ s_2 \end{pmatrix} \\
&\xrightarrow{\gamma_2^{-1}} \begin{pmatrix} v_4 \\ s_4 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} v_4 \\ s_5 \end{pmatrix} \\
&\xrightarrow{\gamma_3} \begin{pmatrix} v_7 \\ s_7 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} v_7 \\ s_8 \end{pmatrix} \\
&\xrightarrow{\gamma_4} \begin{pmatrix} v_6 \\ s_6 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} v_6 \\ s_7 \end{pmatrix} \\
&\xrightarrow{\gamma_3^{-1}} \begin{pmatrix} v_5 \\ s_5 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} v_5 \\ s_6 \end{pmatrix} \\
&\xrightarrow{\gamma_4^{-1}} \begin{pmatrix} v_8 \\ s_8 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} v_8 \\ s_9 \end{pmatrix} \\
&\xrightarrow{\gamma_5} \begin{pmatrix} v_0 \\ s_10 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} v_0 \\ s_1 \end{pmatrix}
\end{aligned}
$$

which gives elliptic cycle

$$\mathcal{E}_1 : v_0 \to v_4 \to v_3 \to v_2 \to v_4 \to v_7 \to v_6 \to v_5 \to v_8$$

which has angle sum $\mathrm{sum}(\mathcal{E}_1) = 9 \times \frac{\pi}{9} = \pi$. Hence the elliptic cycle has order $m_1 = 2$ (so that $m \times \mathrm{sum}(\mathcal{E}) = 2\pi$).
[When this was an exam question, lots of people took $m = 1$.]
We also have the elliptic cycle

$$\begin{pmatrix} v_9 & s_9 \end{pmatrix} \xrightarrow{\gamma_5} \begin{pmatrix} v_9 & s_10 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} v_9 & s_9 \end{pmatrix}.$$

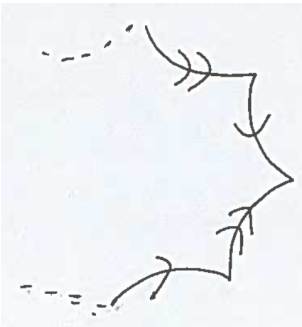
Thus we have elliptic cycle

$$\mathcal{E}_2 : v_9$$

which has angle sum $\mathrm{sum}(\mathcal{E}_2) = \pi/9$. This has order $m_2 = 18$.
Hence the Elliptic Cycle Condition holds for both elliptic cycles. Hence Poincaré's Theorem says that $\gamma_1, \ldots, \gamma_5$ generate a Fuchsian group Γ.

(ii) There are two marked points: one given by gluing together the vertices on $\mathcal{E}_1$ to give a marked point of order 2, and the other given by gluing together the vertices on $\mathcal{E}_2$ to give a marked point of order 18.

There are two copies of side-pairing transformations of the following form:



(The other two sides glue together to give one of the marked points.) This suggests that the genus of 2.

[If you don't look this 'stare-at-it' method, then you could think about Euler's formula. The surface $\mathbb{H}/\Gamma$ will have a triangulation with $V = 2$ vertices (the number of elliptic cycles), $E = 10/2 = 5$ edges and $F = 1$ face. Hence $2 - 2g = \chi = V - E + F = 2 - 5 + 1 = -2$, so $g = 2$.

Hence $\mathrm{sig}(\Gamma) = (2; 2, 18)$.

$\mathbb{H}/\Gamma$ looks like the following: