What did we do last time?

- Stated that, for a Fuchsian group, the side-pairing txs generate the group.

  (Eg: $z \mapsto z+1$, $z \mapsto -1/z$ generate $PSL(2,\mathbb{Z})$.)

- Discussed how to define abstract groups via generators & relations

What will we do today?

- State Poincaré's Thm: this allows us to construct a large number of Fuchsian groups.

Recall: (vertex, side)-chasing:

$$\begin{pmatrix} v_0 \\ s_0 \end{pmatrix} \xrightarrow{\gamma_1} \begin{pmatrix} v_1 \\ s_1 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} v_1 \\ *s_1 \end{pmatrix}$$

side-pairing tx asuc. to s_0

replace s_1 with the other side with an end pt at v_1

$$\xrightarrow{\gamma_2} \begin{pmatrix} v_2 \\ s_2 \end{pmatrix} \xrightarrow{*} \cdots$$

side-pairing tx asuc to $*s_1$

19. Poincaré's Thm: the case of no boundary vertices

Idea: Start with a hyperbolic polygon (not necessarily a Dirichlet polygon) equipped with a set of side-pairing txs.

Q: When do these side-pairing txs generate a Fuchsian group?

Let: D be a convex hyperbolic polygon

Assume: • no vertices of D are on the boundary

• D is equipped with a set G of side-pairing txs

ie $\forall$ sides s of D $\exists\ \gamma_s \in G$ st $\gamma_s(s) = s'$, another side of D (and $\gamma_{s'} = \gamma_s^{-1}$).

• technical hypothesis on half-planes.

Construct elliptic cycles & elliptic cycle txs as before:

$$\begin{pmatrix} v_0 \\ s_0 \end{pmatrix} \xrightarrow{\gamma_1} \begin{pmatrix} v_1 \\ s_1 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} v_1 \\ *s_1 \end{pmatrix}$$

$\gamma_1 =$ side-pairing tx assoc to s_0

$$\xrightarrow{\gamma_2} \begin{pmatrix} v_2 \\ s_2 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} v_2 \\ *s_2 \end{pmatrix}$$

$$\cdots\cdots$$

$$\xrightarrow{\gamma_n} \begin{pmatrix} v_n \\ s_n \end{pmatrix} \xrightarrow{*} \begin{pmatrix} v_n \\ *s_n \end{pmatrix}$$

Choose n least s.t. $(v_n, *s_n) = (v_0, s_0)$.

$\mathcal{E} : v_0 \to v_1 \to \cdots \to v_{n-1}$ = elliptic cycle

$\gamma_n \gamma_{n-1} \cdots \gamma_2 \gamma_1$ = elliptic cycle tx

$\text{sum}(\mathcal{E}) = \angle v_0 + \cdots + \angle v_{n-1}$ = angle sum of $\mathcal{E}$.

<u>Defn</u>. The elliptic cycle ~~E~~ $\mathcal{E}$ satisfies the <u>Elliptic Cycle Condition</u> (ECC) if: $\exists\, m = m(\mathcal{E}) \geq 1$ (an integer) s.t. $m \times \operatorname{sum}(\mathcal{E}) = 2\pi$.

<u>Poincaré's Thm</u>: Let D be a convex hyp. polygon, finitely many sides, no vertices on the boundary.
Assume D is equipped with a set of side-pairing txs $\mathcal{G}$ as above.
Suppose no side of D is paired with itself.
Let $\mathcal{E}_1, \dots, \mathcal{E}_r$ be the elliptic cycles.
Suppose each elliptic cycle satisfies the ECC: $\exists\, m_j \geq 1$ s.t.

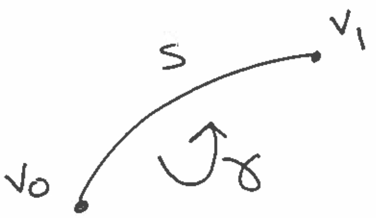
$$m_j \times \operatorname{sum}(\mathcal{E}_j) = 2\pi \qquad 1 \leq j \leq r.$$

Then:
- The subgroup $\Gamma = \langle \mathcal{G} \rangle$ generated by the side-pairing txs is a Fuchsian group.
- D is a fundamental domain for Γ.
- We can give a presentation of Γ in terms of

generators + relations as follows. Regard $\mathcal{G}$, the set of side-pairing txs, as an abstract set of symbols. Each elliptic cycle tx is a word of symbols chosen from $\mathcal{G} \cup \mathcal{G}^{-1}$. For each j, let γ_j be an elliptic cycle tx corresponding to $\mathcal{E}_j$. Then we have the relation $\gamma_j^{m_j} = e$. We have the presentation

$$\Gamma \cong \langle \gamma_s \in \mathcal{G} \mid \gamma_1^{m_1} = \gamma_2^{m_2} = \dots = \gamma_r^{m_r} = e \rangle$$

<u>Rmk</u> The hypothesis that no side of D is paired with itself can be removed. Suppose s is paired with itself.

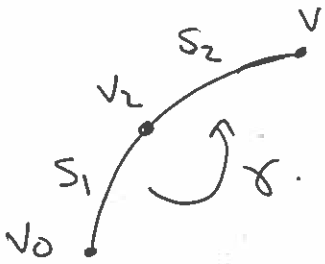


γ maps v_0 to v_1 and v_1 to v_0.

Let v_2 = the mid point of s.

Introduce two new sides $[v_0, v_2] = s_1$, $[v_2, v_1] = s_2$.

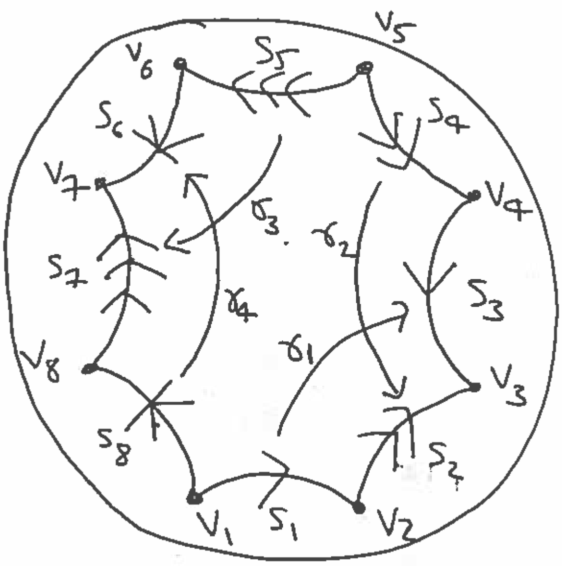
and a new vertex at v_2



Then γ maps s_1 to s_2 and s_2 to s_1.

Note: $\angle v_2 = \pi$.

<u>Example</u> Let D be a regular hyp. octagon with internal angle $\pi/4$.



$$\begin{pmatrix} v_1 \\ s_1 \end{pmatrix} \xrightarrow{\gamma_1} \begin{pmatrix} v_4 \\ s_3 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} v_4 \\ s_4 \end{pmatrix} \xrightarrow{\gamma_2} \begin{pmatrix} v_3 \\ s_2 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} v_3 \\ s_3 \end{pmatrix} \longrightarrow \cdots$$

One can show there is <u>one</u> elliptic cycle:

$$\mathcal{E}: v_1 \to v_4 \to v_3 \to v_2 \to v_5 \to v_8 \to v_7 \to v_6$$

elliptic cycle tx: $\gamma_4^{-1}\gamma_3^{-1}\gamma_4\gamma_3\gamma_2^{-1}\gamma_1^{-1}\gamma_2\gamma_1$

$$\mathrm{sum}(\mathcal{E}) = 8 \times \frac{\pi}{4} = 2\pi.$$

The ECC holds with $m = 1$ $(m \times \mathrm{sum}(\mathcal{E}) = 2\pi)$.

By Poincaré, $\gamma_1, \to \gamma_4$ generate a Fuchsian group Γ & it has a presentation $(a = \gamma_1, b = \gamma_2, c = \gamma_3, d = \gamma_4)$

$$\langle a, b, c, d \mid (d^{-1}c^{-1}dcb^{-1}a^{-1}ba)^1 = e \rangle.$$

What did we do last time?

- Stated Poincaré's Thm (case of no boundary vertices)

  This gives a way of constructing a large number of examples of Fuchsian groups.

Poincaré's Thm (no boundary vertices, & slightly abbreviated)

Suppose
- D is a convex hyp. poly. equipped with a set of side-pairing txs
- no side of D is paired with itself
- all elliptic cycles satisfy the ECC.

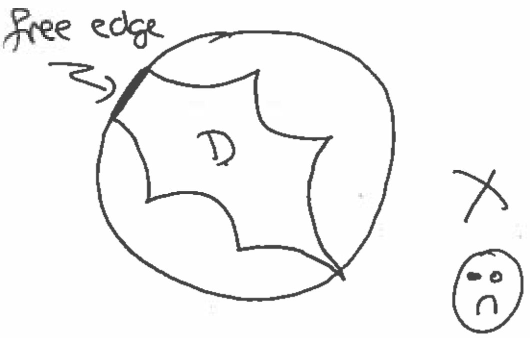
Then
- side-pairing txs generate a Fuchsian group Γ
- D is a fundamental domain for Γ
- one can give a presentation for Γ in terms of generators & relations.

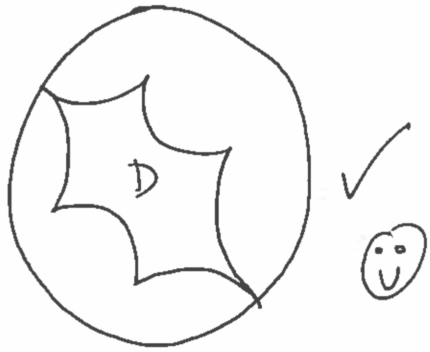
What will we do today?

- Poincaré's Thm (the case of boundary vertices).

20 Poincaré's Thm: the case of boundary vertices

Let D convex hyp. polygon. We will allow D to have vertices on the boundary, but assume that no arcs of the boundary are edges of D (such edges are called <u>free</u> <u>edges</u>).





Assume D is a convex hyp. polygon as above, equipped with a set of side-pairing txs $\mathcal{G}$ (+ the same technical hypotheses as in the previous lecture).

Note: if v is a boundary vertex & is an endpoint of side s, then $\gamma_s(v)$ is also a boundary vertex (as Möbius txs map the boundary to the boundary).

Use (vertex, side)-pair chasing as before. Let v_0 = boundary vertex

$$\begin{pmatrix} v_0 \\ s_0 \end{pmatrix} \xrightarrow{\gamma_1} \begin{pmatrix} v_1 \\ s_1 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} v_1 \\ *s_1 \end{pmatrix}$$

γ_1 = side-pairing tx assoc to s_0.

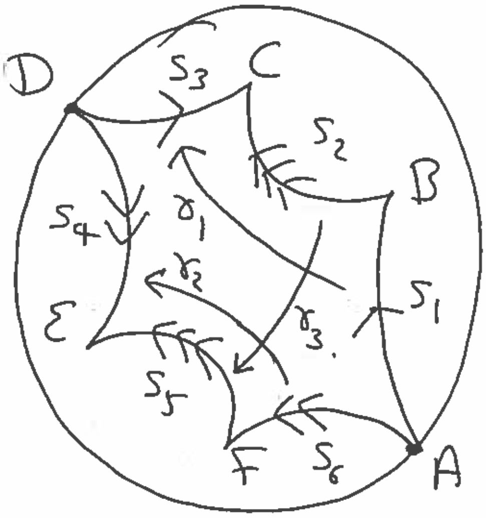
$$\xrightarrow{\gamma_2} \begin{pmatrix} v_2 \\ s_2 \end{pmatrix} \xrightarrow{*} \cdots$$

$$\cdots \xrightarrow{\gamma_n} \begin{pmatrix} v_n \\ s_n \end{pmatrix} \xrightarrow{*} \begin{pmatrix} v_n \\ *s_n \end{pmatrix} = \begin{pmatrix} v_0 \\ s_0 \end{pmatrix}$$

n least.

$\mathcal{P} = v_0 \to v_1 \to v_2 \to \cdots \to v_{n-1}$ is called a <u>parabolic cycle</u>.

$\gamma_{(v_0, s_0)} = \gamma_n \cdots \gamma_2 \gamma_1$ is called a <u>parabolic cycle tx</u>.



$$\begin{pmatrix} A \\ s_1 \end{pmatrix} \xrightarrow{\gamma_1} \begin{pmatrix} D \\ s_3 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} D \\ s_4 \end{pmatrix} \xrightarrow{\gamma_2^{-1}} \begin{pmatrix} A \\ s_6 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} A \\ s_1 \end{pmatrix}$$

Parabolic cycle: $A \to D$

Parabolic cycle tx: $\gamma_2^{-1}\gamma_1$

Elliptic cycle: $B \to F \to \mathcal{E} \to C$

Elliptic cycle tx: $\gamma_1 \gamma_3^{-1} \gamma_2 \gamma_3$

The parabolic cycles & elliptic cycles partition the vertices of D.

Note: If v is a boundary vertex & $\gamma_{(v,s)}$ is a parabolic cycle ~~tx~~ then $\gamma_{(v,s)}(v) = v$.

<u>Defn</u> $\mathcal{P}$ satisfies the <u>Parabolic Cycle Condition (PCC)</u> if:

$\gamma_{(v,s)}$ is either parabolic or the identity.

(Equivalently $\tau(\gamma_{(v,s)}) = 4$.).

<u>Thm</u> (Poincaré's Thm: boundary vertices, no free edges)

Suppose: D is a convex hyp polygon, finitely many sides, no free edges. Assume D is equipped with a set $\mathcal{G}$ of side-pairing txs s.t. no side is paired with itself.

Let $\mathcal{E}_1, \ldots, \mathcal{E}_r$ be the elliptic cycles, $\mathcal{P}_1, \ldots, \mathcal{P}_s$ be the parabolic cycles.

Suppose each elliptic cycle satisfies the ECC: $\exists\, m_j \geq 1$ st $m_j \times \text{sum}(\mathcal{E}_j) = 2\pi$.

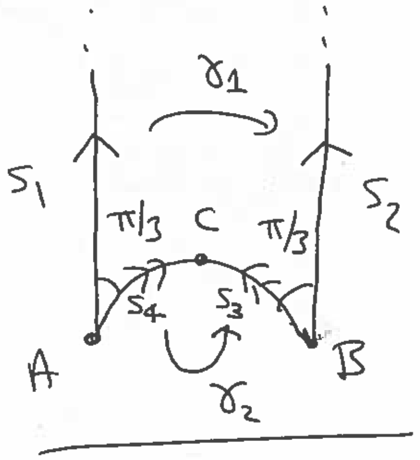
Suppose each parabolic cycle satisfies the PCC.

Then: (1) the side-pairing txs generate a Fuchsian group Γ
(2) D is a fundamental domain for Γ.
(3) Γ has a presentation in terms of generators & relations as follows. Regard G as a set of abstract symbols & each elliptic cycle tx as a word in symbols from $G \cup G^{-1}$. For each j, $1 \le j \le r$, we have the relation $\gamma_j^{m_j} = e$.
(γ_j = elliptic cycle tx assoc. to $\mathcal{E}_j$)

Then $\Gamma \cong \langle \gamma_s \in G \mid \gamma_1^{m_1} = \cdots = \gamma_r^{m_r} = e \rangle$.

<u>Rmk</u> Parabolic cycles don't give any relations - we just need the PCC to hold.

<u>Example</u>: Use Poincaré's Thm to find a presentation of $PSL(2,\mathbb{Z})$.



$A = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ $\quad B = \frac{1}{2} + \frac{i\sqrt{3}}{2}$.

$\gamma_1(z) = z+1$, $\quad \gamma_2(z) = \frac{-1}{z}$.

Introduce a new vertex $C = i$ so that no side is paired with itself

$$\begin{pmatrix} A \\ s_1 \end{pmatrix} \xrightarrow{\gamma_1} \begin{pmatrix} B \\ s_2 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} B \\ s_3 \end{pmatrix} \xrightarrow{\gamma_2^{-1}} \begin{pmatrix} A \\ s_4 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} A \\ s_1 \end{pmatrix}$$

Elliptic cycle $\mathcal{E}_1 : A \to B$
Elliptic cycle tx: $\gamma_2^{-1}\gamma_1$
$\text{sum}(\mathcal{E}_1) = \frac{2\pi}{3}$.

ECC holds with $m_1 = 3$.

$$\begin{pmatrix} C \\ s_4 \end{pmatrix} \xrightarrow{\gamma_2} \begin{pmatrix} C \\ s_3 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} C \\ s_4 \end{pmatrix}$$

Elliptic cycle $\mathcal{E}_2 : C$
Elliptic cycle tx: γ_2
$\text{sum}(\mathcal{E}_2) = \pi$.

ECC holds with $m_2 = 2$. (4)

$$\begin{pmatrix} \infty \\ s_1 \end{pmatrix} \xrightarrow{\gamma_1} \begin{pmatrix} \infty \\ s_2 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} \infty \\ s_1 \end{pmatrix}.$$

Parabolic cycle $\mathcal{P}_1$: ∞

Parabolic cycle tx: γ_1.

PCC holds as γ_1 is a translation & so is parabolic.

By Poincaré's Thm, γ_1, γ_2 generate a Fuchsian group Γ (which we already know is $PSL(2,\mathbb{Z})$) and Γ has a presentation: $(a = \gamma_1, \ b = \gamma_2)$,

$$\Gamma \cong \langle a, b \mid (b^{-1}a)^3 = b^2 = e \rangle.$$

Solutions to Week 10 worksheet

(i) $\Gamma \subset \text{Möb}(\mathbb{H})$ is a Fuchsian group if it is a discrete subgroup of $\text{Möb}(\mathbb{H})$.

(ii) We know that Möbius transformations map geodesics to geodesics. So to show that γ_1, γ_2 pair the claimed sides, we only need show that γ_1, γ_2 map the end-points of the respective sides to each other, i.e. we need to check:

(a) $\gamma_1(-(1+\cos\theta) + i\sin\theta) = 1 + \cos\theta + i\sin\theta$,
(b) $\gamma_1(\infty) = \infty$,
(c) $\gamma_2(0) = 0$,
(d) $\gamma_2((1+\cos\theta) + i\sin\theta) = -(1+\cos\theta) + i\sin\theta$.

(a) follows easily as $\gamma_1(-(1+\cos\theta)+i\sin\theta) = -(1+\cos\theta) + i\sin\theta + 2 + 2\cos\theta = 1 + \cos\theta + i\sin\theta$.

(b) follows as γ_1 is a translation and so fixes ∞.

(c) follows as $\gamma_2(0) = 0/(-0+1) = 0$.

(d) follows as

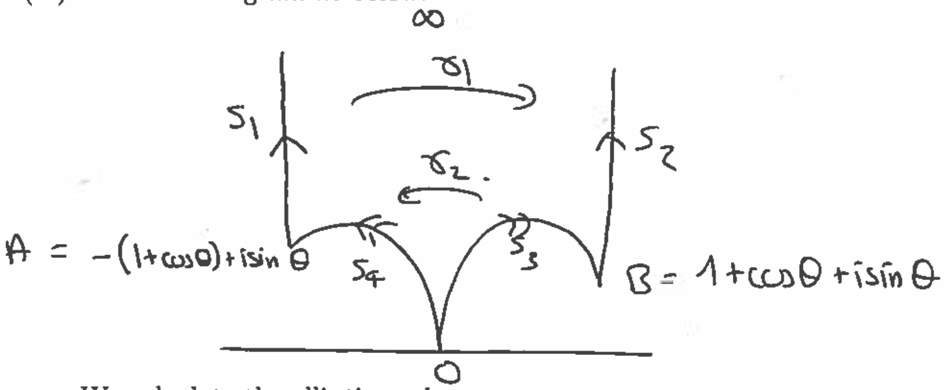
$$\begin{aligned}
\gamma_2((1+\cos\theta)+i\sin\theta) &= \frac{(1+\cos\theta)+i\sin\theta}{-(1+\cos\theta)-i\sin\theta+1} \\
&= \frac{(1+\cos\theta)+i\sin\theta}{-\cos\theta - i\sin\theta} \times \frac{-\cos\theta+i\sin\theta}{-\cos\theta+i\sin\theta} \\
&= -(1+\cos\theta)+i\sin\theta.
\end{aligned}$$

expanding out the bracket and recalling that $\cos^2\theta + \sin^2\theta = 1$.

(iii) An elliptic cycle $\mathcal{E}$ satisfies the ECC if there exists an integer $m \geq 1$ such that $m \times \text{sum}(\mathcal{E}) = 2\pi$.

A parabolic cycle $\mathcal{P}$ satisfies the PCC if a corresponding parabolic cycle transformation is either parabolic or the identity.

(iv) Label the diagram as below:



We calculate the elliptic cycle:

$$\begin{pmatrix} A \\ s_1 \end{pmatrix} \xrightarrow{\gamma_1} \begin{pmatrix} B \\ s_2 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} B \\ s_3 \end{pmatrix} \xrightarrow{\gamma_2} \begin{pmatrix} A \\ s_4 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} A \\ s_1 \end{pmatrix}.$$

This gives:

- elliptic cycle $\mathcal{E} : A \to B$
- elliptic cycle transformation $\gamma_2\gamma_1$
- angle sum $\mathrm{sum}(\mathcal{E}) = 2\theta$.

Thus the ECC holds iff there exists $m \geq 1$ such that $2\pi = m\mathrm{sum}(\mathcal{E}) = 2m\theta$, i.e. iff $\theta = \pi/m$ for some integer $m \geq 1$. The question assumes that $\theta \in (0, \pi)$, so we require $m \geq 2$.

We have the parabolic cycle

$$\begin{pmatrix} \infty \\ s_1 \end{pmatrix} \xrightarrow{\gamma_1} \begin{pmatrix} \infty \\ s_2 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} \infty \\ s_1 \end{pmatrix}.$$

This gives:

- parabolic cycle $\mathcal{P}_2 : 0$
- parabolic cycle transformation γ_2.

Note that $\gamma_2(z) = z/(-z+1)$ is in normalised form and $\tau(\gamma_2) = (1+1)^2 = 4$. Hence γ_2 is parabolic and so the PCC holds.

We have the parabolic cycle

$$\begin{pmatrix} 0 \\ s_3 \end{pmatrix} \xrightarrow{\gamma_1} \begin{pmatrix} 0 \\ s_4 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} 0 \\ s_3 \end{pmatrix}.$$

This gives:

- parabolic cycle $\mathcal{P}_2 : \infty$
- parabolic cycle transformation γ_1.

Note that $\gamma_1(z)$ is a translation. Hence γ_1 is parabolic and so the PCC holds.

Let $\theta = \pi/m$ for an integer $m \geq 2$. Then Poincaré's Theorem holds and γ_1, γ_2 generate a Fuchsian group Γ.

As there is one elliptic cycle, there is just one relation. Thinking in terms of abstract symbols, let $a = \gamma_1$, $b = \gamma_2$. We have

$$\Gamma = \langle a, b \mid (ba)^m = e \rangle.$$

(v) Let $\theta = \pi/2$ so that $m = 2$ in the above. We have

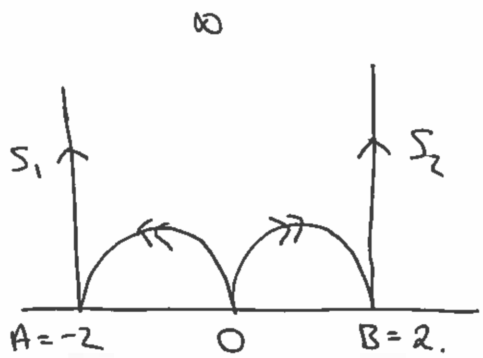
$$\gamma_1(z) = z + 2, \quad \gamma_2(z) = \frac{z}{-z+1}.$$

By (iii), these should satisfy the relation $(\gamma_2\gamma_1)^2 = \mathrm{id}$; we can verify this directly as follows. Recall that composing Möbius transformations is given by multiplying the corresponding matrices together. We have

$$\left(\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right)^2 = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Either by recalling that scalar multiples of a matrix give the same Möbius transformation or simply noting that the above matrix gives Möbius transformation $(-z + 0)/(0z - 1) = z$, we see that $\gamma_2\gamma_1)^2 = \mathrm{id}$.

(vi) The free group $\mathcal{F}_2$ has two generators and no relations. Relations come from elliptic cycles. Thus we should expect to be able to construct it by removing the elliptic cycle in the construction in (iii). Consider what happens in the 'limit' as $\theta \to 0$ or, equivalently, if the vertices A, B are on the real axis (at $A = -2, B = 2$, respectively). We would have the picture as below.



The same calculation as above show that we have the following parabolic cycles:

- $\mathcal{P}_1 = 0$ with parabolic cycle transformation γ_2,
- $\mathcal{P}_2 = \infty$ with parabolic cycle transformation γ_1.

(In both of these cases the PCC continues to hold for the same reasons as in (iii).)

However, A, B are now also boundary vertices and so must also be on a parabolic cycle. The same (vertex,side)-pair chasing as in (iii) shows that we have the parabolic cycle

- $\mathcal{P}_3 = A \to B$ with parabolic cycle transformation $\gamma_2\gamma_1$.

We need to check that the PCC holds. Note that $\gamma_2\gamma_1$ has matrix

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ -1 & -3 \end{pmatrix}$$

which is normalised and has $\tau(\gamma_2\gamma_1) = (1-3)^2 = 4$. Hence $\gamma_2\gamma_1$ is parabolic and so the PCC holds.

Hence, by Poincaré's Theorem, γ_1, γ_2 generate a Fuchsian group Γ. There are two generators (a, b, corresponding to γ_1, γ_2), but no relations (as there are no elliptic cycles). Hence $\Gamma = \langle a, b \rangle$, the free group on 2 generators.

[Adapted from a previous exam question.]

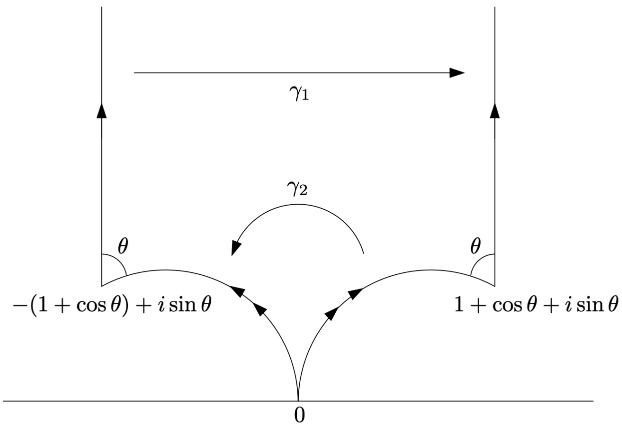
(i) Let $\Gamma \subset \mathrm{Mob}(\mathbf{H})$. What does it mean to say that Γ is a Fuchsian group?

Γ is a ____________________ ____________________ of $\mathrm{Mob}(\mathbf{H})$.

(ii) Consider the hyperbolic quadrilateral illustrated below. Here $\theta \in (0,\pi)$. Let

$$\gamma_1(z) = z + 2 + 2\cos\theta, \qquad \gamma_2(z) = \frac{z}{-z+1}.$$

Show (by considering end points of the sides) that γ_1, γ_2 pair the sides as indicated.



(iii) An elliptic cycle $\mathscr{E}$ satisfies the elliptic cycle condition if: ____________________________

A parabolic cycle $\mathscr{P}$ satisfies the parabolic cycle condition if: ________________________

(iv) Label the vertices and sides of the quadrilateral. Show that there is **one** elliptic cycle and **two** parabolic cycles. Show that the PCC holds for all $\theta \in (0,\pi)$. Determine conditions on θ such that the ECC holds. Hence determine conditions on θ that ensure that γ_1, γ_2 generate a Fuchsian group Γ.

In each case when Γ is a Fuchsian group, give a presentation of Γ in terms of generators and relations.

$$\Gamma = \langle \qquad\qquad \mid \qquad\qquad \rangle. \qquad (*)$$

(v) Consider the special case when $\theta = \pi/2$ so that $\gamma_1(z) = z+2$, $\gamma_2(z) = z/(-z+1)$ (which do generate a Fuchsian group). Check by explicit calculation that γ_1, γ_2 satisfy the relation(s) that you obtained in (*).

(vi) By modifying the construction in (iii) above, show that $\gamma_1(z) = z+4$, $\gamma_2(z) = \dfrac{z}{-z+1}$ generate $\mathscr{F}_2$, the free group on 2 generators.