What did we do last time?

- Saw how to find side-pairing txs & how to record them in a diagram
- "(vertex, side)-chasing"

  Γ = Fuchsian group, $D(p)$ = Dirichlet poly equipped with side-pairing txs.

$$\begin{pmatrix} v_0 \\ s_0 \end{pmatrix} \xrightarrow[\text{side-pairing tx assoc. to } s_0]{\gamma_1} \begin{pmatrix} v_1 \\ s_1 \end{pmatrix} \xrightarrow[\substack{\text{replace } s_1 \\ \text{with the other} \\ \text{side of } D(p) \text{ with} \\ \text{end point at } v_1}]{*} \begin{pmatrix} v_1 \\ *s_1 \end{pmatrix}$$

$$\xrightarrow{\gamma_2} \begin{pmatrix} v_2 \\ s_2 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} \cdots \end{pmatrix}$$

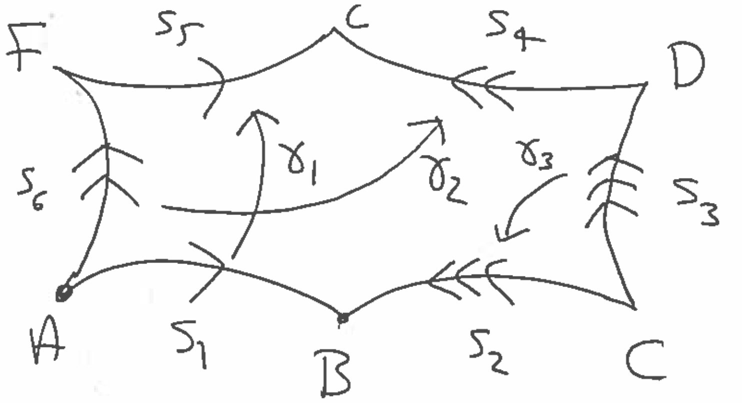
$$\vdots$$

$$\xrightarrow{\gamma_n} \begin{pmatrix} v_n \\ s_n \end{pmatrix} \xrightarrow{*} \begin{pmatrix} v_n \\ *s_n \end{pmatrix}$$

Choose n least s.t. $(v_n, *s_n) = (v_0, s_0)$

Elliptic cycle: $\mathcal{E}: v_0 \to v_1 \to \cdots \to v_{n-1}$

Elliptic cycle tx: $\gamma_{(v_0, s_0)} = \gamma_n \cdots \gamma_1$.



$$\begin{pmatrix} A \\ s_1 \end{pmatrix} \xrightarrow{\gamma_1} \begin{pmatrix} F \\ s_5 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} F \\ s_6 \end{pmatrix}$$

$$\xrightarrow{\gamma_2} \begin{pmatrix} E \\ s_4 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} E \\ s_5 \end{pmatrix}$$

$$\xrightarrow{\gamma_1^{-1}} \begin{pmatrix} B \\ s_1 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} B \\ s_2 \end{pmatrix}$$

$$\xrightarrow{\gamma_3^{-1}} \begin{pmatrix} D \\ s_3 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} D \\ s_4 \end{pmatrix}$$

$$\xrightarrow{\gamma_2^{-1}} \begin{pmatrix} A \\ s_6 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} A \\ s_1 \end{pmatrix}.$$

Elliptic cycle: $A \to F \to E \to B \to D$

Elliptic cycle tx: $\gamma_2^{-1}\gamma_3^{-1}\gamma_1^{-1}\gamma_2\gamma_1$

$$\begin{pmatrix} C \\ s_3 \end{pmatrix} \xrightarrow{\gamma_3} \begin{pmatrix} C \\ s_2 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} C \\ s_3 \end{pmatrix}$$

Elliptic cycle: C

Elliptic cycle tx: γ_3.

Rmks (1) Suppose we started at $(v, *s)$ instead of (v,s). Then we go through the same vertices but in the reverse order and $\gamma_{(v,*s)} = \gamma_{(v,s)}^{-1}$

(2) Suppose we started at $(v_i, *s_i)$ instead of (v_0, s_0). Then we go through a cyclic permutation of the same vertices & γ.

$$\gamma_{(v_i,*s_i)} = \gamma_i\gamma_{i-1}\cdots\gamma_1\gamma_n\cdots\gamma_{i+1}$$

$$= (\gamma_i\gamma_{i-1}\cdots\gamma_1)(\gamma_n\cdots\gamma_{i+1})(\gamma_i\cdots\gamma_1)(\gamma_i\cdots\gamma_1)^{-1}$$

$$= (\gamma_i\cdots\gamma_1)\,\gamma_{(v_0,s_0)}\,(\gamma_i\cdots\gamma_1)^{-1}$$

– so $\gamma_{(v_i,*s_i)}$, $\gamma_{(v_0,s_0)}$ are conjugate.

(3). We have $\gamma_{(v,s)}(v) = v$. So $\gamma_{(v,s)}$ has a fixed point at $v \in \mathbb{H}$. So $\gamma_{(v,s)}$ is either elliptic or the identity.

<u>Defn</u> If $\gamma_{(v,s)}$ is the identity then we call the elliptic cycle an <u>accidental cycle</u>.

<u>Order</u> Let $\gamma \in \text{Möb}(\mathbb{H})$ (or $\text{Möb}(\mathbb{D})$). We say γ has <u>finite order</u> if $\exists\, n \geq 1$ st $\gamma^n = id$. The least such n is called the <u>order of γ</u>.

<u>Example</u> $\gamma(z) = e^{2\pi i \theta} z \in \text{Möb}(\mathbb{D})$.

γ has finite order $\iff$ $\theta = p/q$ (lowest terms)

q = order of γ.

<u>Prop</u> Let Γ be a Fuchsian group. Let $\gamma \in \Gamma$ be elliptic. Then γ has finite order, ie $\exists\, m \geq 1$ st $\gamma^m = id$.

<u>Rmks</u> (1) Suppose γ has order m. Then γ^{-1} also has order m.

(2) Suppose γ_1 has order m and γ_2 is conjugate to γ_1.

Then
$$\gamma_2^m = (g^{-1}\gamma_1 g)^m = g^{-1}\gamma_1 \cancel{g}\,\cancel{g^{-1}}\gamma_1 \cancel{g} \cdots \cancel{g^{-1}}\gamma_1 g$$
$$= g^{-1}\gamma_1^m g = g^{-1} g = id.$$

So γ_2 has order m.

<u>Defn</u> Let $\gamma_{(v,s)} \in \Gamma$ be an elliptic cycle tx.

Then $\gamma_{(v,s)}$ is either elliptic or the identity.

Hence $\gamma_{(v,s)}$ has finite order. By the above remarks, the order of $\gamma_{(v,s)}$ is independent of which (vertex, side)-pair on the elliptic cycle $\mathcal{E}$ we started with.

Define $m(\mathcal{E})$ = order of the elliptic cycle $\mathcal{E}$

$=:$ order of $\gamma_{(v,s)}$.

<u>Defn</u> Let $\mathcal{E} = v_0 \to v_1 \to \cdots \to v_{n-1}$. If v is a vertex let $\angle v$ = internal angle at v. We define the <u>angle sum of $\mathcal{E}$</u> to be

$$\mathrm{sum}(\mathcal{E}) = \angle v_0 + \angle v_1 + \cdots + \angle v_{n-1}.$$

<u>Prop</u> Let Γ be a Fuchsian group. Let D be a Dirichlet polygon with all vertices in $\mathbb{H}$ (or $\mathbb{D}$). Let $\mathcal{E}$ be an elliptic cycle.

Then $$m(\mathcal{E}) \times \mathrm{sum}(\mathcal{E}) = 2\pi.$$

<u>18. Generators & relations</u>

<u>Defn</u> Let Γ be a group. A subset $S = \{\gamma_1, \ldots, \gamma_n\} \subset \Gamma$ is a <u>set of generators</u> if every element of Γ can be written as a product of elements in S and their inverses.

<u>Examples</u> ① $\mathbb{Z}$ is generated by $S = \{+1\}$ as

$$n = \underbrace{+1 + 1 + \cdots + 1}_{n \text{ times}} \qquad -n = \underbrace{-1 - 1 - \cdots - 1}_{n \text{ times}}.$$

2. $\mathbb{Z}^2 = \{(n,m) \mid n, m \in \mathbb{Z}\}$ is generated by

$$S = \{(0,1), (1,0)\}.$$

3. Let $\omega = e^{2\pi i/p}$ = p^{th} root of unity.

Let $\Gamma = \{1, \omega, \omega^2, \dots, \omega^{p-1}\}$ (group operation is multiplication).

Then Γ is generated by $S = \{\omega\}$.

What did we do last time?

- Γ = Fuchsian group $\to$ D = Dirichlet polygon $\to$ side-pairing txs $\to$ elliptic cycles/ ellip. cycle txs.

- Each elliptic cycle $\mathcal{E}$ has an <u>order</u> $m(\mathcal{E})$ and an <u>angle sum</u> $\text{sum}(\mathcal{E})$.

$$m(\mathcal{E}) \times \text{sum}(\mathcal{E}) = 2\pi.$$

- <u>Generators</u>: Let Γ be a group. $S \subset \Gamma$ is a <u>set of generators</u> if every element of Γ can be written as a product of elements from S and their inverses.

What will we do today?

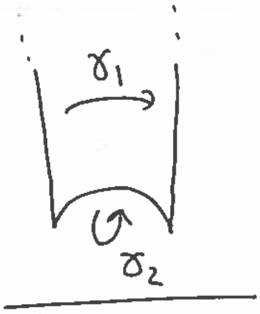
- See how side-pairing txs generate a Fuchsian group
- Talk about how to define groups abstractly using generators & relations.

Rmk A given group Γ may have many different sets of generators.

Example $\mathbb{Z}$ is generated by $\{+1\}$. $\mathbb{Z}$ is also generated by $\{2, 3\}$.

Thm Let Γ be a Fuchsian group, let D be a Dirichlet polygon with $\text{Area}_{\mathbb{H}}(D) < \infty$. Equip D with a set of side-pairing txs. Then the set of side-pairing txs generates Γ.

Example $\Gamma = PSL(2,\mathbb{Z}) = \left\{ \gamma(z) = \dfrac{az+b}{cz+d},\ \begin{matrix} a,b,c,d \in \mathbb{Z} \\ ad-bc=1 \end{matrix} \right\}$.



$\gamma_1(z) = z+1$, $\gamma_2(z) = -\dfrac{1}{z}$. Then $PSL(2,\mathbb{Z})$ is generated by γ_1, γ_2.

Groups defined abstractly by generators & relations

Let S = finite set of symbols. If $a \in S$ is a symbol then we introduce a new symbol a^{-1}. Let $S^{-1} = \{a^{-1} \mid a \in S\}$.

A **word** is a finite concatenation of symbols chosen from $S \cup S^{-1}$ subject to any occurrences of $a^{-1}a$, aa^{-1} as subwords are removed.

Let $W_n = \{\text{all words of length } n\} = \left\{ a_1 \cdots a_n \;\middle|\; \begin{matrix} a_j \in S \cup S^{-1} \\ a_{j\pm1} \neq a_j^{-1} \end{matrix} \right\}$

$W_0 = \{e\}$ $\quad e$ = empty word (= the word with no symbols in it).

If w_n, w_m are words of length n, m respectively, then we can form a new word $w_n w_m$ by concatenation.

Then $w_n w_m$ is a word of length $\leq n+m$.

Eg: $w_n = ab^2a^{-1}b$ $\qquad w_m = b^{-1}a^2ba^{-1}$

$$w_n w_m = ab^2a^{-1}\cancel{b}\,\cancel{b^{-1}}\,a^2ba^{-1}$$

$$= ab^2a^{-1}a^2ba^{-1}$$

$$= ab^2aba^{-1}$$

Note: we cannot move symbols past each other!

Suppose S has k symbols in it.

Let $\mathcal{F}_k = \bigcup_{n=0}^{\infty} W_n$ = free group on k generators,

with group operation given by concatenation of words.

Then this :
- is well-defined (obvious)
- is associative (surprisingly hard!)
- the empty word e is the group identity:

$$we = w = ew$$

- inverses: $w = a_1 \cdots a_n$

$$w^{-1} = a_n^{-1} \cdots a_1^{-1}$$

$$(ww^{-1} = w^{-1}w = e).$$

<u>Defn</u> A <u>relation</u> is a word that we declare to be equal to the identity. We are allowed to cancel any occurrence of a relation if it occurs as a subword.

Let $S = \{a_1, \ldots a_n\}$ be a set of generators.

Let $w_1, \ldots, w_m$ = finite set of words (these will be the relations).

Define $\Gamma = \langle a_1, \ldots, a_n \mid w_1 = \cdots = w_m = e \rangle$ to be the group of all words of symbols from $S \cup S^{-1}$ subject to:

(1) aa^{-1}, $a^{-1}a$ as subwords are replaced by e

(2) any occurrence of $w_1, \ldots, w_m$ as subwords are replaced by e.

Call Γ the group generated by $a_1, \ldots, a_n$ & with relations $w_1, \ldots, w_m$.

<u>Defn</u> A group is <u>finitely presented</u> if it is isomorphic to $\langle a_1, \ldots, a_n \mid w_1 = \cdots = w_m = e \rangle$ for some set of generators & relations. Call this a <u>presentation</u> of the group

<u>Examples</u>

① The free group on k generator $\mathcal{F}_k = \langle a_1, \ldots, a_k \rangle$ is finitely presented.

(2) $\mathbb{Z}$ is finitely presented. $\mathbb{Z}$ is isomorphic to $\langle a \rangle$.

$$\mathbb{Z} \ni n \longleftrightarrow a^n \in \langle a \rangle$$

(3) Let $\omega = e^{2\pi i/p}$ = primitive p^{th} root of 1.

$\Gamma = \{1, \omega, \ldots, \omega^{p-1}\} = \{p^{th}$ roots of $1\}$.

This is finitely presented & is isomorphic to

$$\langle a \mid a^p = e \rangle$$

$$\Gamma \ni \omega^j \longleftrightarrow a^j \in \langle a \mid a^p = e \rangle.$$

(4). $\mathbb{Z}^2 = \{(n,m) \mid n, m \in \mathbb{Z}\}$ is finitely presented. It is isomorphic to

$$\Gamma = \langle a, b \mid a^{-1} b^{-1} a b = e \rangle.$$

Note: $ba = bae = \cancel{ba\, a^{-1} b^{-1}} ab = ab$

So any word $a^{n_1} b^{m_1} a^{n_2} b^{m_2} \ldots a^{n_k} b^{m_k}$ can be written (using the fact $ab = ba$) as $a^{n_1 + \cdots + n_k} b^{m_1 + \cdots + m_k}$

$= a^n b^m$ & a, b commute.

$$\mathbb{Z}^2 \ni (n,m) \longleftrightarrow a^n b^m \in \Gamma.$$

(5). $\langle a, b \mid a^4 = b^2 = (ab)^2 = e \rangle$

$\boxed{(ab)^2 = abab \text{ NOT } a^2 b^2}$

= symmetries of a square

a = rot by $90°$, b = flip in a diagonal.

(6) $\langle a, b \mid a^2 = (ab)^3 = e \rangle$ is $PSL(2, \mathbb{Z})$