

What did we do last time?

Let Γ be a Fuchsian group.

* Procedure for finding a fundamental domain:

- ① Find $p \in \mathbb{H}$ s.t. $\gamma(p) \neq p \quad \forall \gamma \in \Gamma \setminus \{\text{id}\}$
- ② Construct $[p, \gamma(p)]$
- ③ Find $\angle_p(\gamma) = \Gamma$ bisector of $[p, \gamma(p)]$
- ④ $H_p(\gamma) = \text{half-plane determined by } \angle_p(\gamma) \text{ that contains } p$.
- ⑤ $D(p) = \bigcap_{\gamma \in \Gamma \setminus \{\text{id}\}} H_p(\gamma)$. = Dirichlet region
(often a polygon).

What will we do today?

Examples of the above!

~~examples~~

15. Dirichlet polygons - examples

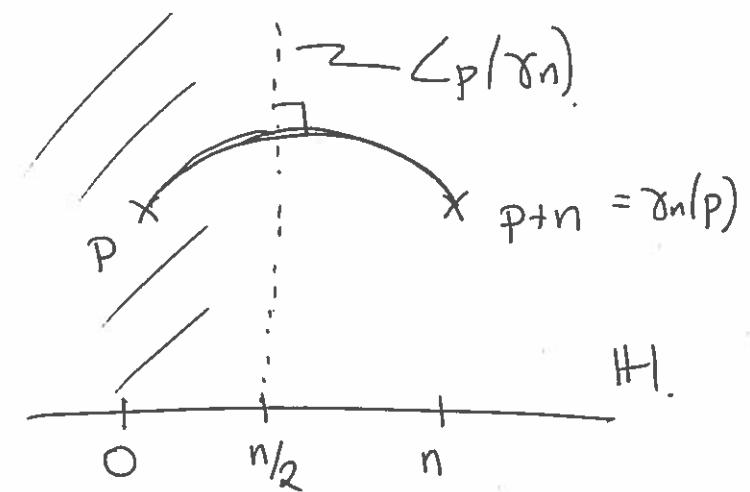
Example 1: $\Gamma = \{\gamma_n \mid \gamma_n(z) = z + n, n \in \mathbb{Z}\}$.

Take $p = i$. Then $\gamma_n(p) \neq p \quad \forall n \neq 0$.

Note $\gamma_n(p) = p + n$.

Suppose $n \geq 1$.

$$\angle_p(\gamma_n) = \left\{ z \in \mathbb{H} \mid \operatorname{Re}(z) = \frac{n}{2} \right\}.$$

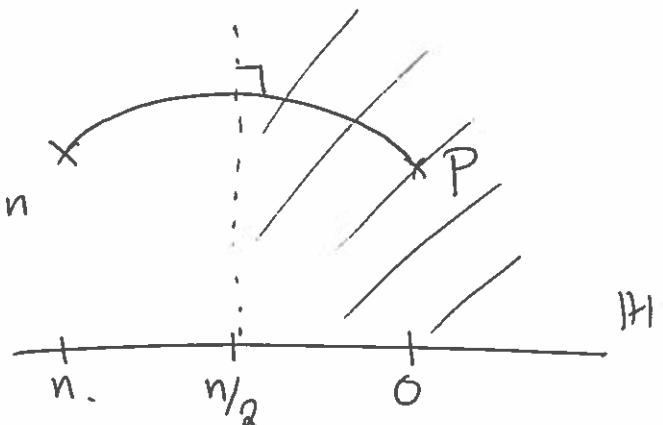


$$\bigcap_{n=1}^{\infty} H_p(\gamma_n) = \left\{ z \in \mathbb{H} \mid \operatorname{Re}(z) < \frac{1}{2} \right\}.$$

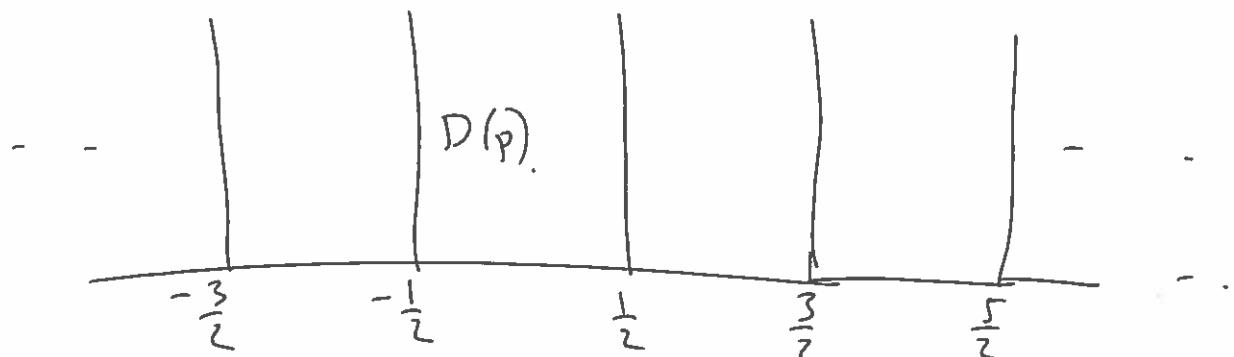
Suppose $n \leq -1$.

$$H_p(\gamma_n) = \left\{ z \in \mathbb{H} \mid \operatorname{Re}(z) > \frac{n}{2} \right\}.$$

$$\bigcap_{n=-1}^{-\infty} H_p(\gamma_n) = \left\{ z \in \mathbb{H} \mid \operatorname{Re}(z) > -\frac{1}{2} \right\}.$$

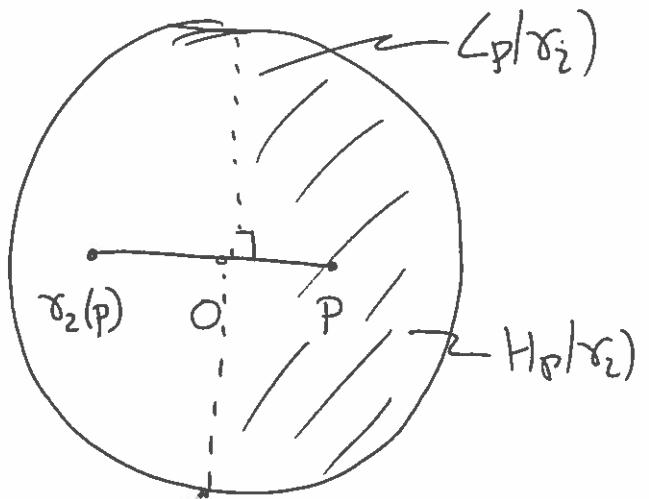
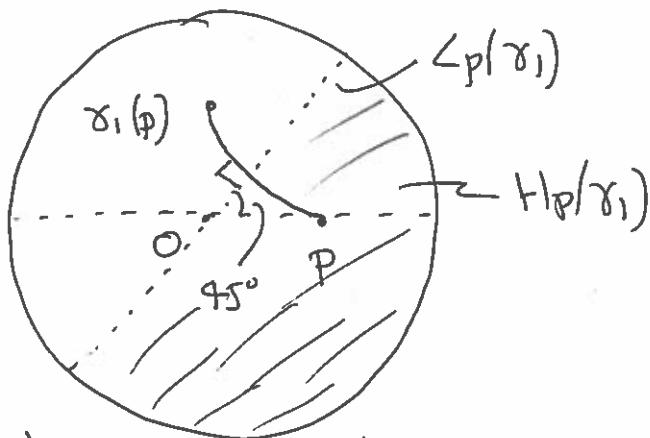


$$D(p) = \bigcap_{n \neq 0} H_p(\gamma_n) = \left\{ z \in \mathbb{H} \mid -\frac{1}{2} < \operatorname{Re}(z) < \frac{1}{2} \right\}.$$

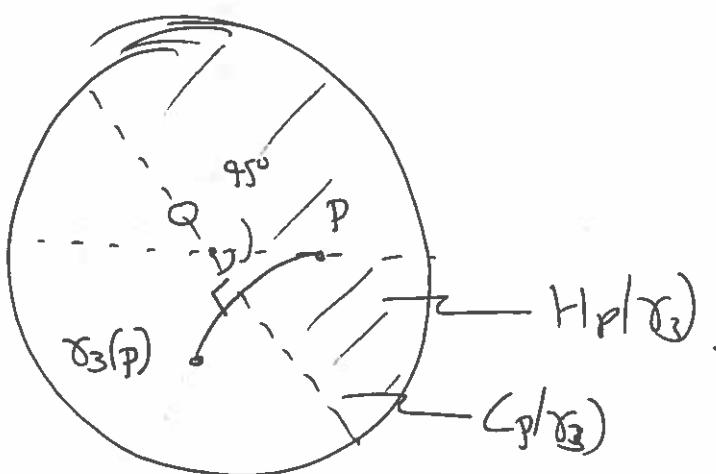


Example 2 $\Gamma = \{ \text{rotations of } D \text{ through angles} \}$ ②
 $0^\circ, 90^\circ, 180^\circ, 270^\circ$
id γ_1 γ_2 γ_3 .

Let $p = 1/2$. Then $\gamma_j(p) \neq p$ $j=1, 2, 3$.



Picture is symmetrical across the diameter at angle $\pi/4$.



$$D(p) = \left\{ z \in D \mid -\frac{\pi}{4} < \arg z < \frac{\pi}{4} \right\}$$

More generally, if $\Gamma = \{ \text{rotations through multiples of } 2\pi/n \}$
 $= \{ \gamma_k(z) = e^{2\pi ik/n} z, k=0, 1, \dots, n-1 \}$

then with $p = 1/2$, $D(p) = \left\{ z \in D \mid -\frac{\pi}{n} < \arg z < \frac{\pi}{n} \right\}$.

Example 3 $\Gamma = \text{PSL}(2, \mathbb{Z}) = \left\{ \gamma(z) = \frac{az+b}{cz+d} \mid a, b, c, d \in \mathbb{Z}, ad-bc=1 \right\}$ ③

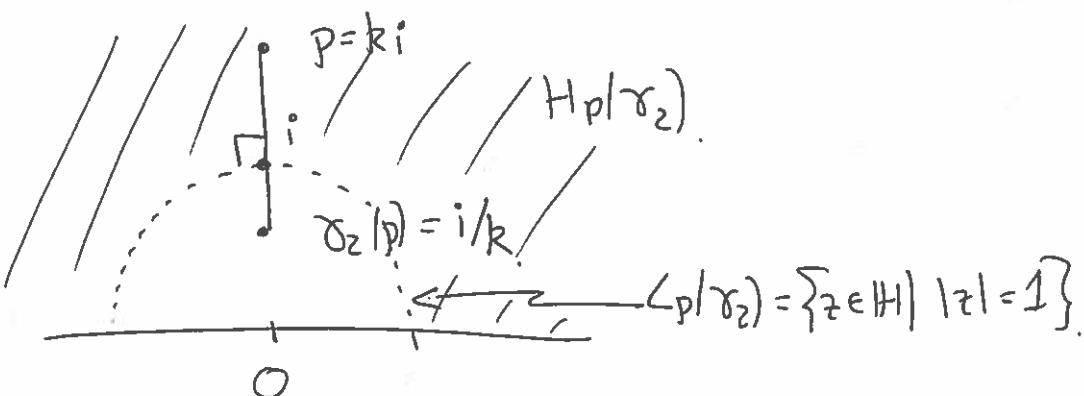
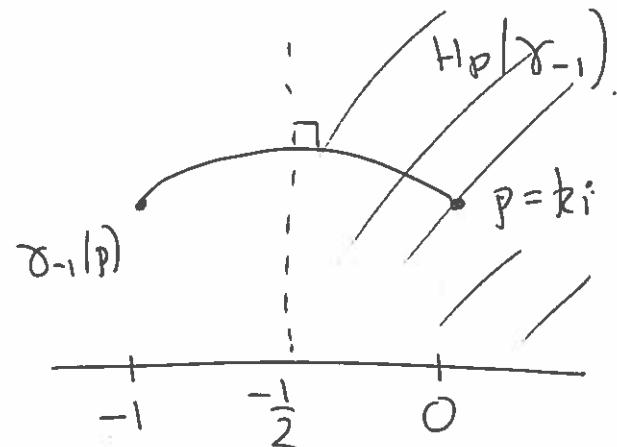
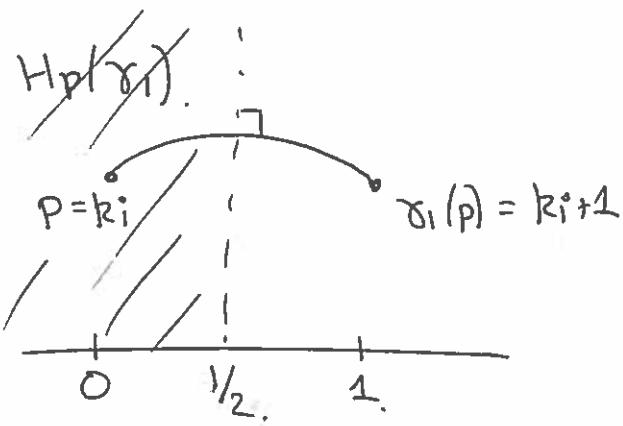
Claim Let $p = ki$, $k > 1$. Then $\gamma(p) \neq p \quad \forall \gamma \in \Gamma \setminus \{\text{id}\}$
 (Note: $p = i$ doesn't work).

Prop $D(p) = \left\{ z \in \mathbb{H} \mid -\frac{1}{2} < \operatorname{Re}(z) < \frac{1}{2}, |z| > 1 \right\}$



Pr. Let $\gamma_1(z) = z+1$, $\gamma_{-1}(z) = z-1$, $\gamma_2(z) = -\frac{1}{z}$.

Then $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$.



Let $F = H_p(\gamma_1) \cap H_p(\gamma_{-1}) \cap H_p(\gamma_2) = \left\{ z \in \mathbb{H} \mid -\frac{1}{2} < \operatorname{Re}(z) < \frac{1}{2}, |z| > 1 \right\}$

Note $D(p) = \bigcap_{\gamma \in \Gamma \setminus \{\text{id}\}} H_p(\gamma) \subset F$.

We claim $D(p) = F$. Suppose for a contradiction that $D(p) \subsetneq F$.

We know $D(p)$ is a fundamental domain, so the images of $D(p)$ under Γ "tile" H . ④

Hence $\exists \gamma \in \Gamma \setminus \{\text{id}\}$ st

$$\gamma(D(p)) \cap F \neq \emptyset$$

Choose $z_0 \in D(p)$,

$$\gamma \in \Gamma \setminus \{\text{id}\} \quad \gamma(z) = \frac{az+b}{cz+d} \quad a, b, c, d \in \mathbb{Z} \quad ad-bc=1.$$

st $\gamma(z_0) \in F$.

Recall $\Im \gamma(z) = \frac{1}{|cz+d|^2} \Im z$.

$$|cz_0+d|^2 = c^2 |z_0|^2 + 2 \operatorname{Re}(z_0) cd + d^2$$

$$\underbrace{c^2}_{>1} \underbrace{d^2}_{-\frac{1}{2} < c < \frac{1}{2}}$$

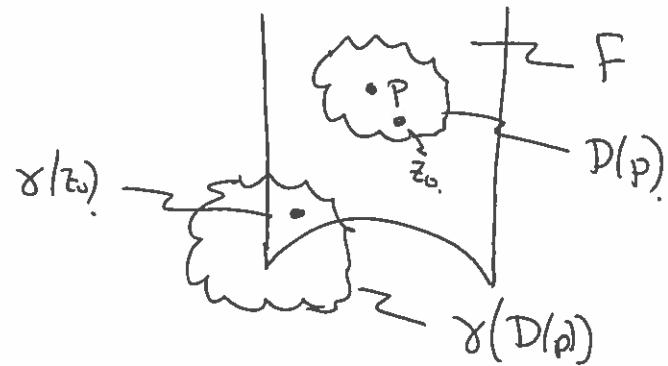
$$\begin{aligned} &> c^2 - |cd| + d^2 \\ &= (\underbrace{|c|-|d|}_{\text{non-neg integer}})^2 + \underbrace{|cd|}_{\text{non-neg integer}} \geq 1. \end{aligned}$$

Hence $|cz_0+d|^2 > 1$. Hence $\Im \gamma(z_0) < \Im z_0$

Repeat the above argument but replace

z_0 by $\gamma(z_0)$ and γ by γ^{-1} .

This shows that $\Im \gamma(z_0) > \Im z_0$.



(5)

Thus in a contradiction,

So $D(P) = F$, ie $F = \underline{\text{L}}$

is a Dirichlet polygon for P

□

What did we do last time?



- Examples of Dirichlet polygons.

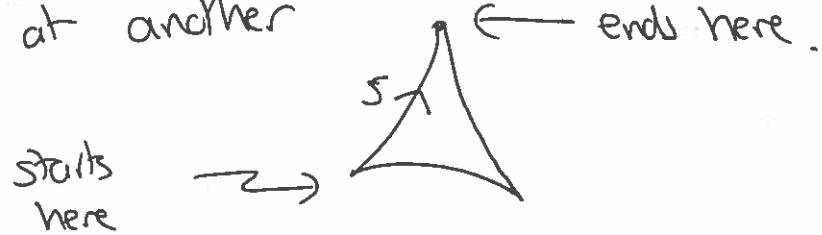
What will we do today?

- \Rightarrow Start looking at how to reconstruct Γ from \mathbb{B} a Dirichlet polygon.

16. Side-pairing txs

①

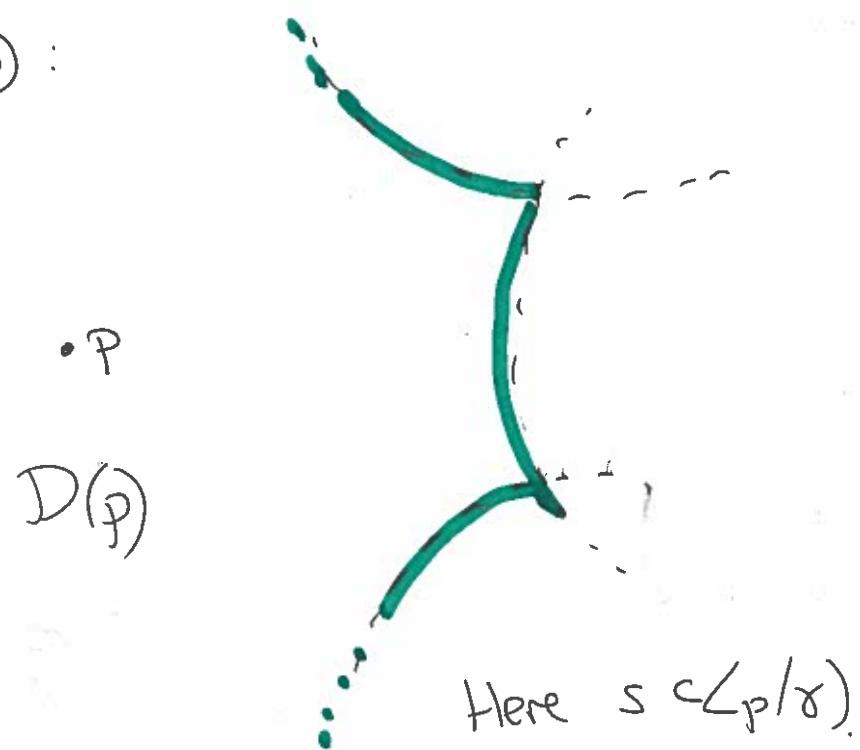
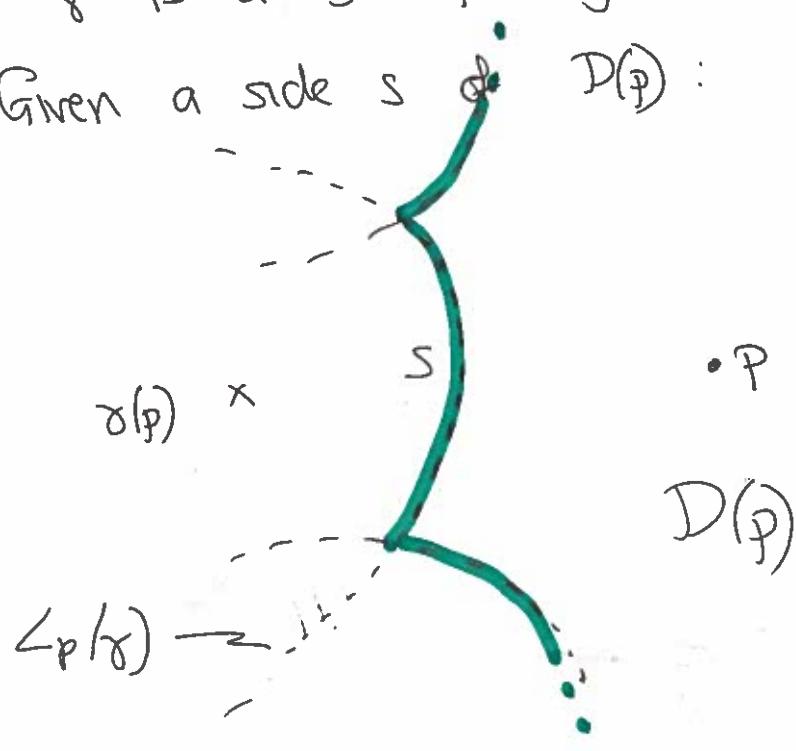
Suppose D is a hyperbolic polygon. A ~~sides~~ side $s \subset H$ is an edge of D equipped with an orientation. So a side start at one vertex and end at another.



Let Γ be a fuchsian group & suppose $D(\Gamma)$ is a Dirichlet polygon (so it has finitely many sides).

Let s be a side of $D(\Gamma)$. Suppose we can find $\gamma \in \Gamma$ s.t. $\gamma(s) = s'$, another side of $D(\Gamma)$. Then $\gamma^{-1} \in \Gamma$ maps s' back to s . We say s, s' are paired and γ is a side-pairing tx.

Given a side s of $D(\Gamma)$:

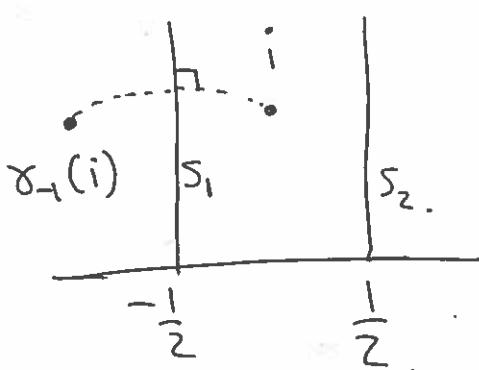


Here $s \subset \angle_p(\gamma)$

Fact. Let $s \subset \angle_p(\gamma)$ is a side of $D(p)$. Then the transformation γ^{-1} maps s to another side s' of $D(p)$. Write $\gamma_s = \gamma^{-1}$. So $\gamma_s(s) = s'$. We have $\gamma_{s'} = \gamma_s^{-1}$.

Example Integer translations $\Gamma = \{\gamma_n(z) = z + n, n \in \mathbb{Z}\}$.

Let $p = i$. Then $D(p) = \{z \in \mathbb{H} \mid -\frac{1}{2} < \operatorname{Re}(z) < \frac{1}{2}\}$



Let $s_1 = \{z \in \mathbb{H} \mid \operatorname{Re}(z) = -1/2\}$

$s_2 = \{z \in \mathbb{H} \mid \operatorname{Re}(z) = 1/2\}$.

$$\gamma_1(z) = z + 1, \quad \gamma_{-1}(z) = z - 1.$$

Then $s_1 = \angle_p(\gamma_{-1})$. So the side-pairing tx associated to side s_1 is $\gamma_{s_1} = (\gamma_{-1})^{-1} \quad \gamma_{s_1}(z) = z + 1$.

~~$\gamma_{s_2} = \gamma_{s_1}^{-1}$~~ Note that $\gamma_{s_1}(s_1) = s_2$.

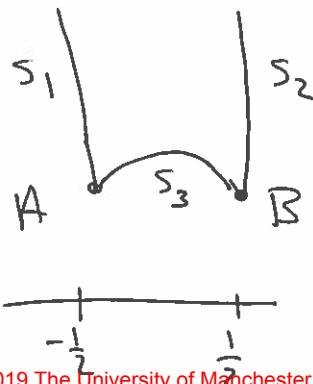
Similarly $s_2 = \angle_p(\gamma_1)$. So $\gamma_{s_2}(z) = \gamma_1(z) = z - 1$

and γ_{s_2} maps s_2 to s_1 . Also note: $\gamma_{s_2} = \gamma_{s_1}^{-1}$

So s_1, s_2 are paired.

Example $\Gamma = \text{PSL}(2, \mathbb{Z})$. Let $p = 2i$, $\tau > 1$.

$$D(p) = \{z \in \mathbb{H} \mid -\frac{1}{2} < \operatorname{Re}(z) < \frac{1}{2}, |z| > 1\}$$



Let $\gamma_1(z) = z + 1, \quad \gamma_{-1}(z) = z - 1$
 $\gamma_2(z) = -1/z$.

Then $s_1 \subset \angle_p(\gamma_{-1}) \quad s_2 \subset \angle_p(\gamma_1)$
 $s_3 \subset \angle_p(\gamma_2)$.

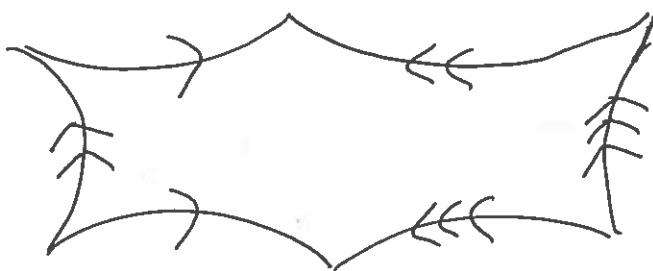
$$\text{So } \gamma_{s_1}(z) = \gamma_1^{-1}(z) = z+1 \quad \gamma_{s_1}(s_1) = s_2. \quad (3)$$

$$\gamma_{s_2}(z) = \gamma_1^{-1}(z) = z-1 \quad \gamma_{s_2}(s_2) = s_1,$$

- ie s_1, s_2 are paired.

Also $\gamma_{s_3}(z) = \gamma_2^{-1}(z) = -\frac{1}{z}$, so s_3 is paired with itself - but note that, as $\gamma_{s_3}(A) = B, \gamma_{s_3}(B) = A$, this side-pairing reverses the orientation.

We record the side-pairing txs on a diagram as follows:



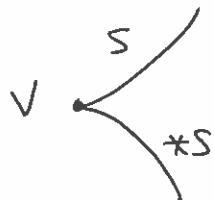
Sides with an equal number of arrow-heads are paired, & the side-pairing preserves the orientation of the arrows.

17. Elliptic cycles

Let Γ be a Fuchsian group, $D = D(\Gamma) =$ Dirichlet polygon with finitely many sides. Assume all the vertices of D are in \mathbb{H}^1 . Equip D with a set of side-pairing txs.

Let (v, s) be a (vertex, side)-pair, ie v is an endpoint of the side s .

If s is a side with an endpoint at v , let $*s$ be the other side of D with an endpoint at v .



④

Consider the following procedure:

- start at (v_0, s_0) .
- let γ_1 be the side-pairing tx associated to s_0 .
- let $s_1 = \gamma_1(s_0)$, $v_1 = \gamma_1(v_0)$.
- consider $(v_1, *s_1)$ & let γ_2 be the side-pairing tx associated to $*s_1$.
- keep going - - -

$$\begin{array}{ccccc} (v_0) & \xrightarrow{\gamma_1} & (v_1) & \xrightarrow{*} & (v_1 \\ & & s_1 & & *s_1) \\ & \xrightarrow{\gamma_2} & (v_2) & \xrightarrow{*} & (v_2 \\ & & s_2 & & *s_2) \\ & \longrightarrow & \cdots & & \end{array}$$

As there are only finitely many (vertex, side) pairs, we must come back to where we started (eventually..)

Choose n least such that $(v_n, *s_n) = (v_0, s_0)$

We call $\Sigma : v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_{n-1}$
an elliptic cycle.

We call $\gamma_{(v_0, s_0)} = \gamma_n \cdots \gamma_2 \gamma_1$ an elliptic cycle tx

The elliptic cycles partition the net of vertices of D .