What did we do last time?

- Defined Fuchsian groups (= discrete subgroups of Möb($\mathbb{H}$) or Möb($\mathbb{D}$))

  [Examples: • $\{$integer translations, $\gamma_n(z) = z+n, n\in\mathbb{Z}\}$
  • $\{$dilations by powers of 2, $\gamma_n(z) = 2^n z, n\in\mathbb{Z}\}$
  • Modular group $PSL(2,\mathbb{Z})$.]

We will see that Fuchsian groups are the symmetry groups of tilings of the hyperbolic plain.

What will we do today?

- Introduce fundamental domains (these will be the "tiles")

Coursework test

- You can collect your coursework test paper from the T&L office reception.

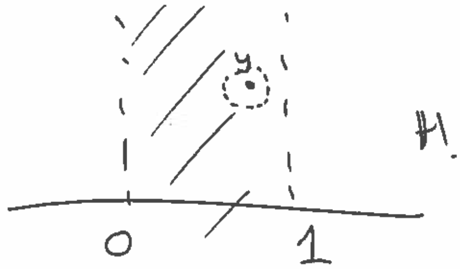
13 Fundamental domains

$\mathcal{Y} \subset \mathbb{H}$ is <u>open</u> if: $\forall y \in \mathcal{Y}$ $\exists \varepsilon > 0$ s.t. $B_\varepsilon(y) = \{z \in \mathbb{H} \mid d(z,y) < \varepsilon\}$ is contained in $\mathcal{Y}$.

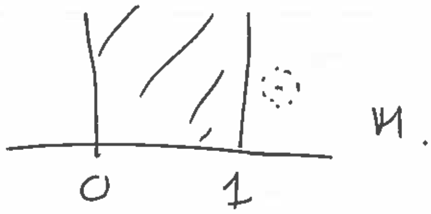
$\mathcal{Y} \subset \mathbb{H}$ is <u>closed</u> if $\mathbb{H} \setminus \mathcal{Y}$ is open.

<u>Examples</u>

1. $\mathcal{Y} = \{z \in \mathbb{H} \mid 0 < \mathrm{Re}(z) < 1\}$ is open.



2. $\mathcal{Y} = \{z \in \mathbb{H} \mid 0 \leq \mathrm{Re}(z) \leq 1\}$ is closed (because the complement is open).



3. $\mathcal{Y} = \{z \in \mathbb{H} \mid 0 < \mathrm{Re}(z) \leq 1\}$ is neither open nor closed.

For us, any set defined by strict ineqs will be open & any set defined by weak ineqs will be closed.

<u>Defn</u> Let $\mathcal{Y} \subset \mathbb{H}$. The <u>closure of $\mathcal{Y}$</u>, $cl(\mathcal{Y})$, is the smallest closed set that contains $\mathcal{Y}$.

<u>Example</u> If $\mathcal{Y} = \{z \in \mathbb{H} \mid 0 < \mathrm{Re}(z) < 1\}$ then
$$cl(\mathcal{Y}) = \{z \in \mathbb{H} \mid 0 \leq \mathrm{Re}(z) \leq 1\}.$$

For us, we can take the closure of a set by replacing strict inequalities by weak inequalities.

<u>Defn</u> Let Γ be a Fuchsian group. An open subset $F \subset \mathbb{H}$ (or $\mathbb{D}$) is a <u>fundamental domain</u> for Γ if

(1) $\bigcup_{\gamma \in \Gamma} \gamma(\mathrm{cl}(F)) = \mathbb{H}$ (or $\mathbb{D}$)

(2) $\gamma_1(F) \cap \gamma_2(F) = \emptyset$ if $\gamma_1, \gamma_2 \in \Gamma$ $\gamma_1 \neq \gamma_2$.

<u>Rmk</u> One can check $\gamma(\mathrm{cl}(F)) = \mathrm{cl}(\gamma(F))$.

<u>Examples</u> ① $\Gamma = \{\gamma_n \mid \gamma_n(z) = z+n,\ n \in \mathbb{Z}\}$

$F = \{z \in \mathbb{H} \mid 0 < \mathrm{Re}(z) < 1\}$

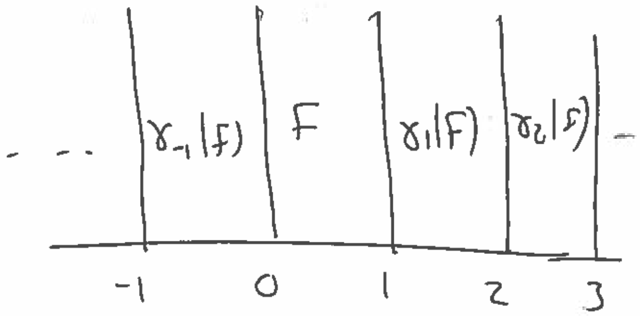
$\gamma_n(F) = \{z \in \mathbb{H} \mid n < \mathrm{Re}(z) < n+1\}$

$\mathrm{cl}(\gamma_n(F)) = \{z \in \mathbb{H} \mid n \leq \mathrm{Re}(z) \leq n+1\}$

Then F is open. $\gamma_n(F) \cap \gamma_m(F) = \emptyset$ if $n \neq m$.

$$\bigcup_{n \in \mathbb{Z}} \mathrm{cl}(\gamma_n(F)) = \mathbb{H}.$$

So F is a fundamental domain for Γ.



② $\Gamma = \{\gamma_n \mid \gamma_n(z) = 2^n z,\ n \in \mathbb{Z}\}$

$F = \{z \in \mathbb{H} \mid 1 < |z| < 2\}$.

$\gamma_n(F) = \{z \in \mathbb{H} \mid 2^n < |z| < 2^{n+1}\}$

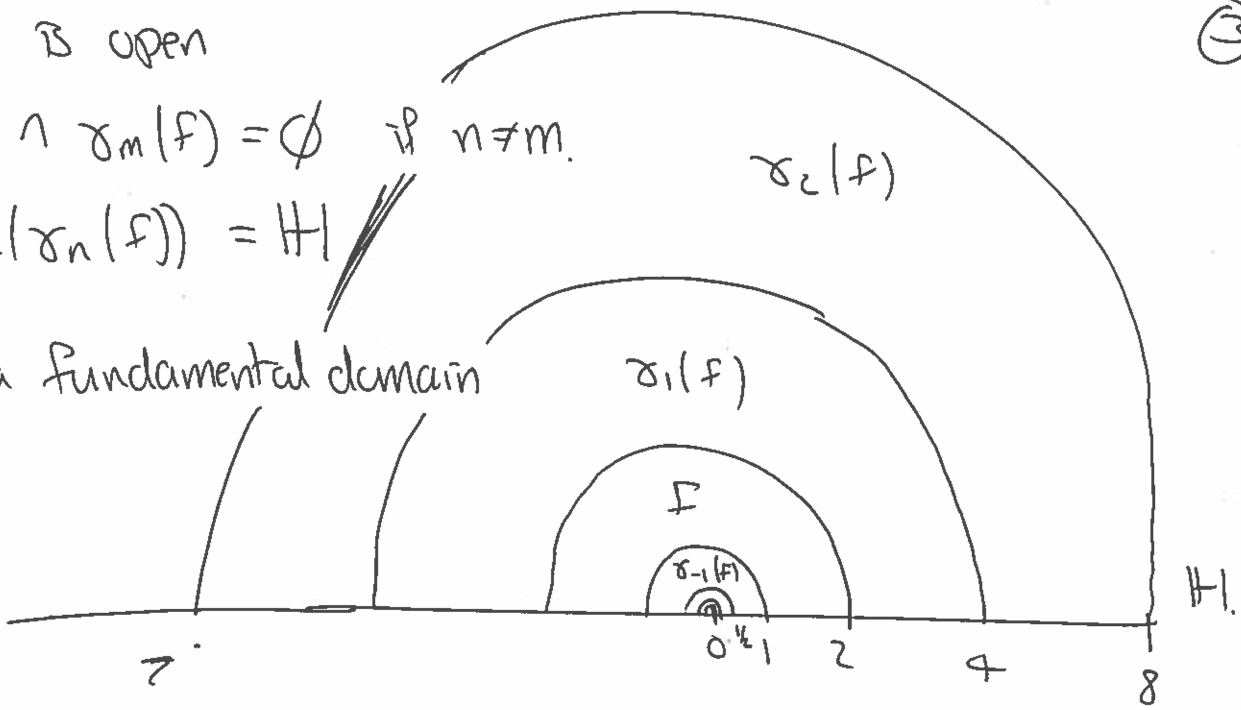
$\mathrm{cl}(\gamma_n(F)) = \{z \in \mathbb{H} \mid 2^n \leq |z| \leq 2^{n+1}\}$.

Then: F is open

$\gamma_n(F) \cap \gamma_m(F) = \emptyset$ if $n \neq m$.

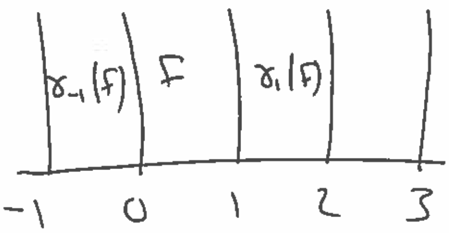
$\bigcup_{n\in\mathbb{Z}} cl(\gamma_n(F)) = \mathbb{H}$

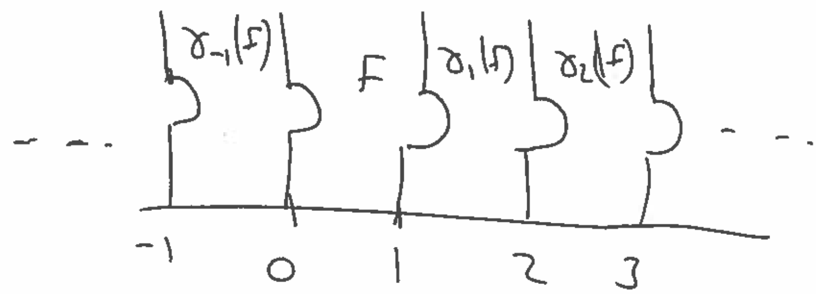
So F is a fundamental domain for Γ.



Fix a Fuchsian group Γ. Then, in general, there are lots of fundamental domains.

Eg: $\Gamma = \{$integer translations$\}$





"Proposition" Let Γ be a Fuchsian group. Let F_1, F_2 be two fundamental domains for Γ. Then

$$\mathrm{Area}_{\mathbb{H}}(F_1) = \mathrm{Area}_{\mathbb{H}}(F_2).$$

Rmk Actually we need extra technical hypotheses here:

(1) F_1, F_2 to be measurable, (2) $\mathrm{Area}_{\mathbb{H}}\, \partial F_i = 0$ $i = 1,2$.

(here $\partial F = cl(F) \setminus F$)

Rmk Area satisfies: $\mathrm{Area}_{\mathbb{H}}\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mathrm{Area}_{\mathbb{H}}(A_n)$.

Pf of Proposition

$$\mathrm{Area}_{\mathbb{H}}\, F_1 = \mathrm{Area}_{\mathbb{H}}\, \mathrm{cl}(F_1) \quad (\text{as } \mathrm{Area}_{\mathbb{H}}\, \partial F_1 = 0).$$

$$\begin{aligned}
\mathrm{cl}(F_1) &= \mathrm{cl}(F_1) \cap \mathbb{H}. \\
&= \mathrm{cl}(F_1) \cap \Big(\bigcup_{\gamma \in \Gamma} \gamma(\mathrm{cl}(F_2)) \Big). \\
&\supset \mathrm{cl}(F_1) \cap \bigcup_{\gamma \in \Gamma} \gamma(F_2) \\
&= \bigcup_{\gamma \in \Gamma} \underbrace{\big(\mathrm{cl}(F_1) \cap \gamma(F_2) \big)}_{\text{are pairwise disjoint as } F_2 \text{ is a fund. dom.}}.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Area}_{\mathbb{H}}\, F_1 &= \mathrm{Area}_{\mathbb{H}}\, \mathrm{cl}(F_1) \\
&\geq \mathrm{Area}_{\mathbb{H}} \Big(\bigcup_{\gamma \in \Gamma} \big(\mathrm{cl}(F_1) \cap \gamma(F_2) \big) \Big). \\
&= \sum_{\gamma \in \Gamma} \mathrm{Area}_{\mathbb{H}} \big(\mathrm{cl}(F_1) \cap \gamma(F_2) \big) && \text{(area of a pairwise disjoint union is the sum of the areas)}. \\
&= \sum_{\gamma \in \Gamma} \mathrm{Area}\; \gamma\big(\gamma^{-1}(\mathrm{cl}(F_1)) \cap F_2 \big). && \text{~~as Möbius t(x) are Area preserving.~~} \\
&= \cancel{A} \sum_{\gamma \in \Gamma} \mathrm{Area}\; \gamma^{-1}(\mathrm{cl}(F_1)) \cap F_2 && \text{as Möbius tx} \text{ are Area preserving} \\
&\geq \mathrm{Area}_{\mathbb{H}} \bigcup_{\gamma \in \Gamma} \gamma^{-1}(\mathrm{cl}(F_1)) \cap F_2 && \big(\cancel{\mathrm{Area}\, \textstyle\sum A_n \Rightarrow A}\big) \\
& && \big(\textstyle\sum \mathrm{Area}\, A_n \geq \mathrm{Area} \bigcup A_n \big) \\
&= \mathrm{Area}_{\mathbb{H}} \Big(\bigcup_{\gamma \in \Gamma} \gamma(\mathrm{cl}(F_1)) \Big) \cap F_2
\end{aligned}$$

$= \mathrm{Area}_{\mathbb{H}}\ \mathbb{H} \cap F_2$ as F_1 is a fund dom. (5)

$= \mathrm{Area}_{\mathbb{H}}\ F_2$.

So $\mathrm{Area}_{\mathbb{H}}\ F_1 \geq \mathrm{Area}_{\mathbb{H}}\ F_2$. Interchange F_1, F_2 to get $\mathrm{Area}_{\mathbb{H}}\ F_2 \geq \mathrm{Area}_{\mathbb{H}}\ F_1$. Hence $\mathrm{Area}_{\mathbb{H}} F_1 = \mathrm{Area}_{\mathbb{H}} F_2$ $\square$

What did we do last time?

- Introduced fundamental domains:

  Let Γ be a Fuchsian group. An open set $F \subset \mathbb{H}$ is a fund. dom if:

  (1) $\bigcup_{\gamma \in \Gamma} \gamma(cl(F)) = \mathbb{H}$

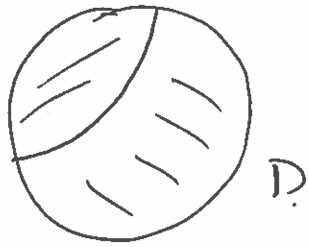
  (2) $\gamma_1(F) \cap \gamma_2(F) = \emptyset \quad \gamma_1, \gamma_2 \in \Gamma \quad \gamma_1 \neq \gamma_2$.

What will we do today?

- Find an algorithm that, for a given Fuchsian group Γ, will give us a fund. dom.

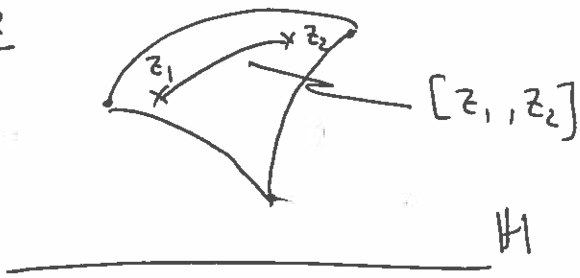
14. Dirichlet polygon: the construction

Let C be a geodesic in $\mathbb{H}$ (or $\mathbb{D}$). Then C divides $\mathbb{H}$ (or $\mathbb{D}$) into two regions. These regions are called half-planes.

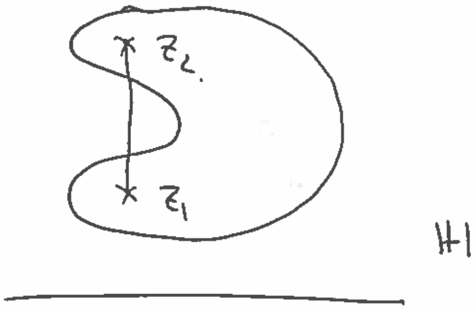


<u>Defn</u> Let $A \subset \mathbb{H}$. We say A is <u>convex</u> if: $\forall z_1, z_2 \in A$ we have $[z_1, z_2] =$ the arc of geodesic from z_1 to z_2 $\subset A$.

<u>Example</u>

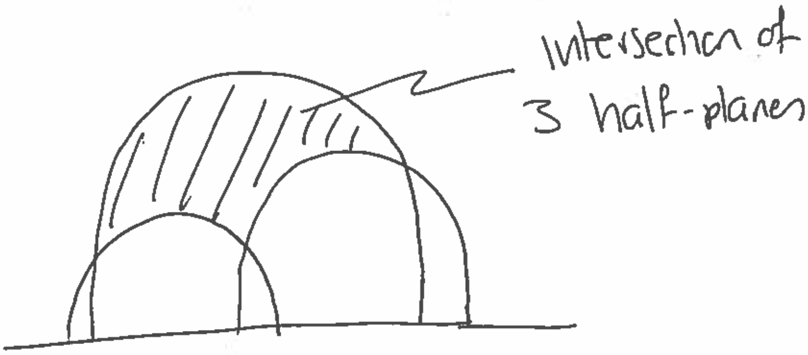


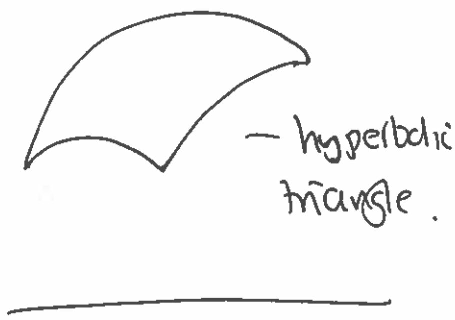
is convex.



not convex.

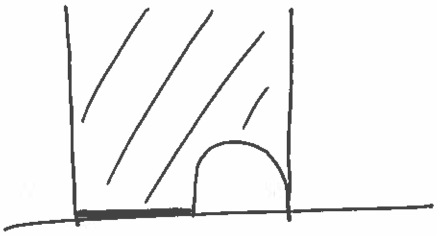
<u>Defn</u> A <u>convex hyperbolic polygon</u> is the intersection of finitely many half-planes.



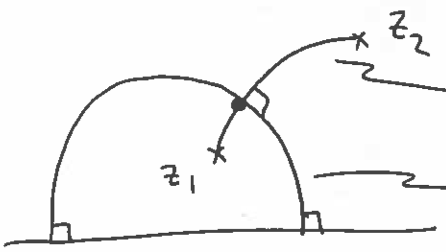


<u>Rmk</u> A convex hyperbolic polygon is a convex set.

<u>Rmk</u> With this defn, we allow arcs of $\partial\mathbb{H}$ to appear as "edges" of a polygon.



<u>$\perp^r$ bisectors</u> The <u>$\perp^r$ bisector</u> of $[z_1, z_2]$ is the unique geodesic that passes through the hyperbolic midpoint of ~~$[z_1$~~ $[z_1, z_2]$ at right-angles.

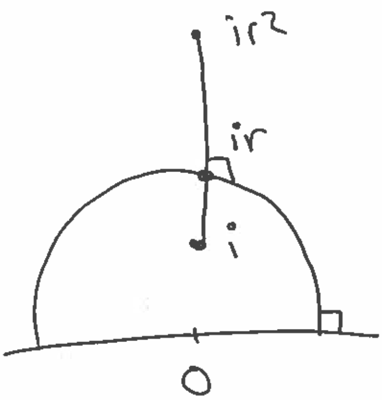


<u>Prop</u> Let $z_1, z_2 \in \mathbb{H}$. The set

$$\{z \in \mathbb{H} \mid d_{\mathbb{H}}(z, z_1) = d_{\mathbb{H}}(z, z_2)\}$$

is the $\perp^r$ bisector of $[z_1, z_2]$.

<u>Pf</u> Apply a Möbius tx so that $[z_1, z_2]$ lies along the imag. axis, $z_1 = i$, $z_2 = ir^2$ (for some $r > 0$).



Recall that if $0 < a < b$ then
$d_{\mathbb{H}}(ia, ib) = \log b/a$.
Then $d_{\mathbb{H}}(ir, i) = d_{\mathbb{H}}(ir^2, ir) = \log r$, so the mid-point is at ir.
So the $\perp^r$ bisector is the semi-circle with centre 0 radius r
ie $|z| = r$.

Also note:

$d_{\mathbb{H}}(z,i) = d_{\mathbb{H}}(z,ir^2) \iff \cosh d_{\mathbb{H}}(z,i) = \cosh d_{\mathbb{H}}(z,ir^2)$

$$\iff \cancel{1+}\frac{|z-i|^2}{\cancel{2\operatorname{Im} z}\operatorname{Im} i} = \cancel{1+}\frac{|z-ir^2|^2}{\cancel{2\operatorname{Im} z}\operatorname{Im} ir^2}$$

$$\iff r^2|z-i|^2 = |z-ir^2|^2$$

$$\iff \text{(boring algebra)} \iff |z| = r \qquad \square$$

<u>Technical lemma</u> Let Γ be a Fuchsian group. Then $\exists\, p \in \mathbb{H}$ s.t. $\gamma(p) \neq p \quad \forall \gamma \in \Gamma \setminus \{id\}$.

Let Γ be a Fuchsian group. Choose $p \in \mathbb{H}$ as in the technical lemma so that $\gamma(p) \neq p \quad \forall \gamma \in \Gamma\setminus\{id\}$.

Let $\gamma \in \Gamma \setminus \{id\}$. Look at

$$\{z \in \mathbb{H} \mid d_{\mathbb{H}}(z,p) < d_{\mathbb{H}}(z,\gamma(p))\} = \left\{\begin{array}{l}\text{points that are closer}\\ \text{to } p \text{ than to } \gamma(p)\end{array}\right\}.$$

<u>Defn</u> $D(p) = \{z \in \mathbb{H} \mid d_{\mathbb{H}}(z,p) < d_{\mathbb{H}}(z,\gamma(p)) \ \forall \gamma \in \Gamma \setminus \{id\}\}$

$$= \left\{\begin{array}{l}\text{points that are closer to } p \text{ than to any other}\\ \text{point on the orbit of } p\end{array}\right\}.$$

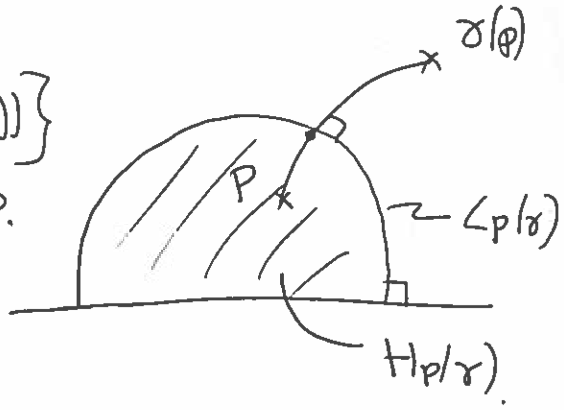
How do we ~~de~~ find $D(p)$ in practice?

1. Choose $p \in \mathbb{H}$ st $\gamma(p) \neq p \quad \forall \gamma \in \Gamma\setminus\{id\}$.
2. Let $\gamma \in \Gamma \setminus \{id\}$. Construct $[p, \gamma(p)]$.
3. $L_p(\gamma) = \perp^r$ bisector of $[p, \gamma(p)]$
   $= \{z \in \mathbb{H} \mid d_{\mathbb{H}}(z,p) = d_{\mathbb{H}}(z,\gamma(p))\}$.

4. $L_p(\gamma)$ divides $\mathbb{H}$ into two half-planes. Let

$$H_p(\gamma) = \{ z \in \mathbb{H} \mid d_{\mathbb{H}}(z, p) < d_{\mathbb{H}}(z, \gamma(p)) \}$$

be the half-plane that contains p.



5. $$D(p) = \bigcap_{\gamma \in \Gamma \setminus \{id\}} H_p(\gamma).$$

Theorem Let Γ be a Fuchsian group. Let p be as in the technical lemma. Then $D(p)$ is a fundamental domain. (We call $D(p)$ a Dirichlet region.)

Moreover, if $\text{Area}_{\mathbb{H}} D(p) < \infty$ then $D(p)$ is a convex hyperbolic polygon. (In this case, we call $D(p)$ a Dirichlet polygon.)

"The best way to learn how to do something is to teach it"
— Albert Einstein, Richard Feynman, plus plenty of other people

"An even better way to learn how to do something is to design an exam on it"
— (less famous) researchers in educational psychology

Background.
Imagine some of your friends have done all of our 1st and 2nd year core material but aren't doing MATH32052. Instead, you have been teaching them all about Hyperbolic Geometry. You've taught them everything that we've covered so far in the course. You decided when you started teaching them that, by the end of your classes, your friends will meet the following *learning outcomes*:

'At the end of the course, students will be able to:
- *calculate the hyperbolic distance between and the geodesic through points in the hyperbolic plane*
- *classify Möbius transformations in terms of their actions on the hyperbolic plane.'*

(These are two of the learning outcomes of this course.)

You've taught your friends all about Möbius transformations (in both the upper half-plane and Poincaré disc models). You've taught them how to compose Möbius transformations together and that they form a group, how to move an arbitrary geodesic to a given geodesic, that they are conformal and area-preserving, etc. You've taught them about parabolic, hyperbolic and elliptic Möbius transformations, how they behave and how to distinguish them, etc. You now want to test your friends to see how well they can meet the above learning outcomes.

Your task.
Working in pairs or in threes, you have to design an exam-style question that assesses how well each of your friends meets the above outcome. Some of your friends are really good at hyperbolic geometry, some are very weak, and some are in-between; your question will need to be able to distinguish between these different abilities.

Your question should have several parts (Q1(i), Q1(ii), etc - the exact number is up to you). When you write your question you will need to allocate marks (say, 10 in total, but again it's up to you) to each part of the question. You'll also need to decide if, and how, you will give partial credit for partially correct answers.

Usually, exam questions build up in terms of difficulty and/or complexity (Q1(i) is easier than Q1(ii), which is easier than Q1(iii), etc). There are several models of exactly what 'difficulty' or 'complexity' mean. One well-used model includes the following levels:

<u>Knowledge</u> (eg: define a concept, state a theorem, list <something>,...)

<u>Comprehension</u> (eg: give an example of <something>, distinguish between similar things,...)

<u>Application</u> (eg: apply a theorem, perform a calculation, prove a theorem,...)

<u>Analysis</u> (eg: compare/contrast two calculations or results, explain relations between concepts, generalise or infer other behaviour,...)

The question you design should have parts that assess your friends' abilities at several of the above levels.

Where do we start?
It's not going to be possible to test everything about Möbius transformations in hyperbolic geometry, so you will need to decide which parts of the material you've covered you will assess and at which level. What do you think are the most important concepts, results, applications? Do you want to start with asking for a definition of something? Or maybe you want to start with asking them to state a theorem? If you ask them to apply a theorem to an example, do you ask them to state that theorem first or do you write it down for them in the question? There's lots of different ways of structuring an exam question!

You could look at some of the exercises in the online notes. Could some parts of your question be adaptations of one or more of these exercises? (Your friends will have seen these exercises: if you decide to use them then you will need to alter them otherwise your question will be too easy!) Maybe the past papers on the course website will suggest some ideas (again, you can't copy them directly: your friends have also seen these past papers!); the question you design will probably be shorter than a question from a past paper though.

TL;DR. What do you want us to do?
In short: write a short exam question that tests how well somebody 'understands' Möbius transformations, together with a mark scheme.

Are you going to use our questions in a real exam?
No, I promise!

How to get started

- Write down a list of all the main definitions and results from the material we've covered on Möbius transformations.
- Find one or more exercises you like and can do. (Maybe one on calculating the trace of a Möbius transformation, or properties of parabolic Möbius transformations, or exploiting the conformality or area-preserving properties of Möbius transformations to prove something, etc.)
- How could you alter or rewrite this exercise so that it is a bit different? Could you changes some of the coefficients in the Möbius transformations you're considering, or change the question so that it's asking about elliptic Möbius transformations rather than, say, parabolic Möbius transformations to get something new?
- Once you've come up with your own 'exercise', how could you turn it into an exam question? You'll need some 'easy' bits and some 'hard' bits (or 'knowledge' and 'analysis', respectively); the exercise you've come up with probably falls under 'application' above.
- Write down a draft exam question. It might be of the form: Q1(i) state a definition/theorem, Q1(ii) do the exercise you thought up, Q1(iii) explain something about how the result in (ii) can be generalised, or how it relates to a theorem in the course.
- Devise a mark scheme.
- Do you think your question is fair? Do you think it assesses the learning outcome above at an appropriate range of levels?