What did we do last time?

0

Used $\tau(\gamma)$ to classify Möbius txs:

$\tau(\gamma) > 4 \iff \gamma$ is hyperbolic

$\tau(\gamma) = 4 \iff \gamma$ is parabolic

$\tau(\gamma) \in [0,4) \iff \gamma$ is elliptic.

Prop Let $\gamma \in \text{Möb}(\mathbb{H})$, $\gamma \neq \text{id}$. The following are equivalent:

(1) γ is parabolic

(2) $\tau(\gamma) = 4$

(3) γ is conjugate to a translation

(4) γ is conjugate to either $z \mapsto z+1$
or $z \mapsto z-1$.

Coursework test

Fri 1^{st} Nov (next week) – check your personalised timetables.

(Univ. approved) calculators are permitted.

Up to & including Sect. 11 in the notes (= today's lecture)

Pf (3) $\Leftrightarrow$ (4) follow from the exercises.

(1) $\Leftrightarrow$ (2) done.

(3) $\Rightarrow$ (1) Suppose γ is conjugate to a translation.
We know translations are parabolic.
Hence γ is parabolic (see last lecture).

(1) $\Rightarrow$ (3) Suppose γ is parabolic with unique fixed point at $\xi \in \partial\mathbb{H}$. Choose $g \in \text{Möb}(\mathbb{H})$ st $g(\xi) = \infty$.
Consider $g\gamma g^{-1}$. Then γ is conjugate to $g\gamma g^{-1}$ as

$$\gamma = g^{-1}(g\gamma g^{-1})g$$

Note $g\gamma g^{-1}(z_0) = z_0 \Leftrightarrow \gamma(g^{-1}(z_0)) = g^{-1}(z_0) \Leftrightarrow g^{-1}(z_0) = \xi$

$$\Leftrightarrow z_0 = g(\xi) = \infty.$$

So $g\gamma g^{-1}$ has a unique fixed pt at ∞.

Claim $g\gamma g^{-1}$ is a translation.

Write $g\gamma g^{-1}(z) = \dfrac{az+b}{cz+d}$. As ∞ is a fixed pt, we have $c = 0$.

So $g\gamma g^{-1}(z) = \dfrac{az+b}{d}$. This has a fixed pt at $\dfrac{b}{d-a}$. As ∞ is the only fixed pt for $g\gamma g^{-1}$ we have $a = d$. Hence $g\gamma g^{-1}(z) = z + b/d$, a translation. $\square$

Hyperbolic Möbius txs

Let γ be hyperbolic, ie γ has 2 fixed pts on $\partial\mathbb{H}$ none in $\mathbb{H}$.

Example $\gamma(z) = kz \quad k \neq 1,\ k > 0.$ — a dilation.

This has fixed pts at $0, \infty$.

Exercise Let $\gamma_1(z) = k_1 z,\ \gamma_2(z) = k_2 z$. Then γ_1, γ_2 are conjugate $\iff k_1 = k_2$ or $k_1 = 1/k_2$.

Proposition Let $\gamma \in \text{Möb}(\mathbb{H})$. The following are equivalent:
(i) γ is hyperbolic, (ii) $\tau(\gamma) > 4$, (iii) γ is conjugate to a dilation.

Pf (i) $\iff$ (ii) done.

(iii) $\Rightarrow$ (i) Dilations are hyperbolic. If γ is conjugate to a hyperbolic tx then γ is hyperbolic.

(i) $\Rightarrow$ (iii) Suppose γ is hyperbolic with fixed pts at $\xi_1, \xi_2 \in \partial\mathbb{H}$. Choose $g \in \text{Möb}(\mathbb{H})$ s.t. $g(\xi_1), g(\xi_2) = 0, \infty$. Consider $g\gamma g^{-1}$. Then $g\gamma g^{-1}$ is conjugate to γ (as $\gamma = g^{-1}(g\gamma g^{-1})g$).

Note: $g\gamma g^{-1}(z_0) = z_0 \iff \gamma(g^{-1}(z_0)) = g^{-1}(z_0)$
$\iff g^{-1}(z_0) = \xi_1, \xi_2 \iff z_0 = g(\xi_1), g(\xi_2)$
$\iff z_0 = 0, \infty.$

Claim $g\gamma g^{-1}$ is a dilation.

Let $g\gamma g^{-1}(z) = \dfrac{az+b}{cz+d}$.

As ∞ is fixed, we have $c = 0$.
As 0 is fixed, we have $b/d = 0$, ie $b = 0$.

Hence $g\gamma g^{-1}(z) = \left(\dfrac{a}{d}\right) z$ — a dilation $\square$

Elliptic Möbius txs

We work in $\mathbb{D}$. Recall that Möbius txs of $\mathbb{D}$ have the form $\gamma(z) = \dfrac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}$ $\quad \alpha, \beta \in \mathbb{C}$, $|\alpha|^2 - |\beta|^2 > 0$

As in Möb($\mathbb{H}$) we can assume wlog that γ is normalised, ie $|\alpha|^2 - |\beta|^2 = 1$.

When γ is normalised, we define $\tau(\gamma) = (\alpha + \bar{\alpha})^2$

<u>Rmk</u> γ is $\left.\begin{array}{l}\text{hyp}\\ \text{parab}\\ \text{ellip}\end{array}\right\} \iff \left.\begin{array}{r}\tau(\gamma) > 4\\ = 4\\ \in [0,4)\end{array}\right\}$.

<u>Example</u> Take $\alpha = e^{i\theta/2}$, $\beta = 0$. Then

$$\gamma(z) = \frac{e^{i\theta/2} z + 0}{0z + e^{-i\theta/2}} = e^{i\theta} z = \text{rotation about } 0 \in \mathbb{D} \text{ through angle } \theta.$$

This has a unique fixed pt at $0 \in \mathbb{D}$, so γ is elliptic

<u>Proposition</u> Let $\gamma \in$ Möb($\mathbb{D}$), $\gamma \neq id$. The following are equivalent

(1) γ is elliptic, (2) $\tau(\gamma) \in [0,4)$, (3) γ is conjugate to a rotation.

<u>Pf</u> <u>(1) $\iff$ (2)</u> as previous lecture.

<u>(3) $\Rightarrow$ (1)</u> Rotations are elliptic. If γ is conjugate to an elliptic tx then γ is itself elliptic.

<u>(1) $\Rightarrow$ (3)</u> Suppose γ has a unique fixed pt at $\zeta \in \mathbb{D}$.

Choose $g \in$ Möb($\mathbb{D}$) s.t. $g(\zeta) = 0$. Consider $g\gamma g^{-1}$.

Then $g\gamma g^{-1}$ is conjugate to γ (as ~~as~~ $\gamma = g^{-1}(g\gamma g^{-1})g$)

Also $g\gamma g^{-1}(z_0) = z_0 \iff \gamma(g^{-1}(z_0)) = g^{-1}(z_0)$ (4)

$$\iff g^{-1}(z_0) = \xi \iff z_0 = g(\xi) = 0.$$

<u>Claim</u> $g\gamma g^{-1}$ is a rotation.

Let $g\gamma g^{-1}(z) = \dfrac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}$. As $g\gamma g^{-1}(0) = 0$, we

have $\dfrac{\beta}{\bar{\alpha}} = 0$, ie $\beta = 0$. Let $\alpha = re^{i\theta}$. Then

$$g\gamma g^{-1}(z) = \frac{re^{i\theta} z}{re^{-i\theta}} = e^{2i\theta} z,$$ a rotation

$\square$

<u>Rmk</u> The Möbius txs of $\mathbb{H}$ given by

$$\gamma(z) = \frac{(\cos \theta/2) z + \sin \theta/2}{(-\sin \theta/2) z + \cos \theta/2}.$$

has a unique fixed pt at $i \in \mathbb{H}$ & is often called a rotation of $\mathbb{H}$ about i.

What did we do last time

Classified parabolic Möbius txs (they look like translations)
hyperbolic Möbius txs (they look like dilations)
elliptic Möbius txs (they look like rotations).

What will we do today?

Introduce Fuchsian groups – discrete subgroups of Möb($\mathbb{H}$).

Coursework test

Fri 1st Nov (next week)
(Univ. approved) calculators are permitted.
Upto & including Sect 11 in the notes (yesterday's lecture).

12. Fuchsian groups

Defn A *Fuchsian group* is a discrete subgroup of $\text{Möb}(\mathbb{H})$ or $\text{Möb}(\mathbb{D})$.

Discreteness Recall (X, d) is a *metric space* if:

(1) $d(x,y) \geq 0$, $d(x,y) = 0 \Leftrightarrow x = y$.

(2) $d(x,y) = d(y,x)$

(3) $d(x,y) \leq d(x,z) + d(z,y)$.

Examples ① $X = \mathbb{R}$ $\quad d(x,y) = |x-y|$

② $X = \mathbb{R}^n$ $\quad d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2}$

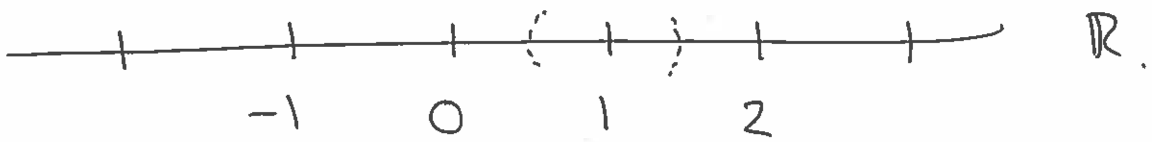
③ $X = \mathbb{H}$ $\quad d = d_{\mathbb{H}}$, $\quad X = \mathbb{D}$ $\quad d = d_{\mathbb{D}}$.

Let (X, d) be a metric space, let $Y \subset X$. A point $y \in Y$ is isolated if: $\exists\, \delta > 0$ s.t. if $y' \in Y$, $y' \neq y$, then $d(y, y') \geq \delta$.

[Equivalently: let $B_\delta(y) = \{z \in X \mid d(z,y) < \delta\}$, then $y \in Y$ is isolated if $\exists\, \delta > 0$ st $B_\delta(y) \cap Y = \{y\}$.]

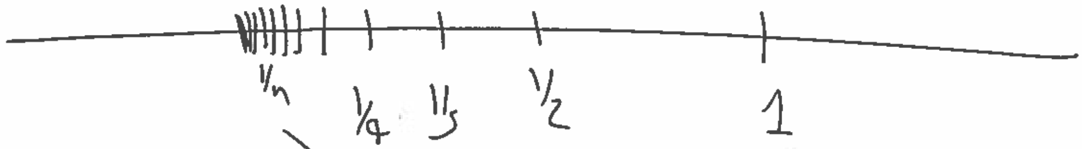
Defn Y is *discrete* if every point in Y is isolated.

Examples ① $X = \mathbb{R}$, $Y = \mathbb{Z}$

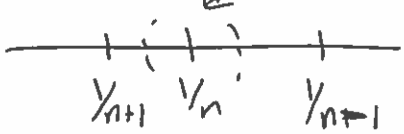


Given $y = 1$, take $\delta = 1/2$. Then there are no other points of Y within distance $1/2$ of .

② $X = \mathbb{R}$, $Y = \{1/n, n \in \mathbb{N}\}$ is discrete.



Take $y = 1/n$



if δ is small enough (note: it will depend on n) then no other point of Y is in here

③ $X = \mathbb{R}$ $Y = \{1/n, n \in \mathbb{N}\} \cup \{0\}$ is not discrete as 0 is not isolated (Given any $\delta > 0$ we can always find $1/n$ s.t. $1/n$ is within distance δ of 0.)

④ $X = \mathbb{R}$, $Y = \mathbb{Q}$. not discrete (no point is isolated).

⑤ X = any set, Y is a finite subset. Then Y is discrete.

A metric on Möb(H).

Q: What does it mean for two Möbius txs to be "near" each other

Example: $\gamma_1(z) = \dfrac{2z+1}{z+1}$, $\gamma_2(z) = \dfrac{2z + 1{\cdot}1}{z+1}$.

— these should be "close"

$\gamma_1(z) = \dfrac{2z+1}{z+1}$ $\gamma_2(z) = \dfrac{4z + 2{\cdot}2}{2z+2}$.

— these should be "close"

$\gamma_1(z) = \dfrac{2z+1}{z+1}$ $\gamma_2(z) = \dfrac{-2z - 1{\cdot}1}{-z-1}$

— these should be "close"

Define $\|(a,b,c,d)\| = \sqrt{a^2+b^2+c^2+d^2}$.

Let $\gamma_i(z) = \dfrac{a_i z + b_i}{c_i z + d_i}$ $\quad a_i, b_i, c_i, d_i \in \mathbb{R}$ $\quad a_i d_i - b_i c_i = 1$

$i = 1, 2$.

Define

$$d_{\text{Möb}(\mathbb{H})}(\gamma_1, \gamma_2) = \min \left\{ \begin{array}{l} \|(a_1,b_1,c_1,d_1) - (a_2,b_2,c_2,d_2)\| \\ \|(a_1,b_1,c_1,d_1) - (-a_2,-b_2,-c_2,-d_2)\| \end{array} \right.$$

Recall a Fuchsian group is a discrete subgroup of $\text{Möb}(\mathbb{H})$ or $\text{Möb}(\mathbb{D})$.

Examples ① Integer translations

$\Gamma = \{ \gamma_n(z) = z + n, \ n \in \mathbb{Z} \}$ is a fuchsian group.

② Dilations by powers of 2

$\Gamma = \{ \gamma_n(z) = 2^n z, \ n \in \mathbb{Z} \}$ is a fuchsian group.

③ Any finite subgroup of Möbius txs is a fuchsian group.

$\Gamma = \{ id, \text{ rotation through } 90°, 180°, 270° \} \subset \text{Möb}(\mathbb{D})$ is a fuchsian group.

Fix $q \in \mathbb{N}$. $\Gamma = \{ \gamma_k(z) = e^{\frac{2\pi i k}{q}} z, \ k = 0, 1, \ldots, q-1 \}$ $\subset \text{Möb}(\mathbb{D})$ is a fuchsian group.

④ If Γ is a fuchsian group & $\Gamma' \subset \Gamma$ is a subgroup then Γ' is a fuchsian group.

⑤ $\Gamma = PSL(2, \mathbb{Z})$ = modular group

$= \left\{ \gamma(z) = \dfrac{az+b}{cz+d}, \ a,b,c,d \in \mathbb{Z} \ \ ad - bc = 1 \right\}$

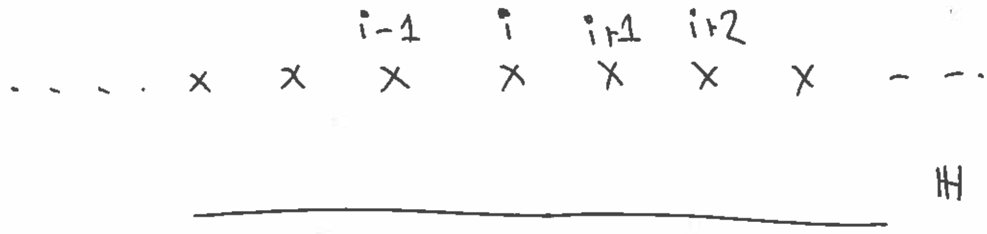
is a fuchsian group.

<u>Orbits</u>. Let Γ be a subgroup of $\text{Möb}(\mathbb{H})$.

Let $z \in \mathbb{H}$. The <u>orbit of z</u> is the set

$$\Gamma(z) = \{\gamma(z) \mid \gamma \in \Gamma\}.$$

<u>Example</u> Γ = integer translations $= \{\gamma_n(z) = z+n \,,\ n \in \mathbb{Z}\}$



$$\Gamma(i) = \{ i+n,\ n \in \mathbb{Z}\}.$$

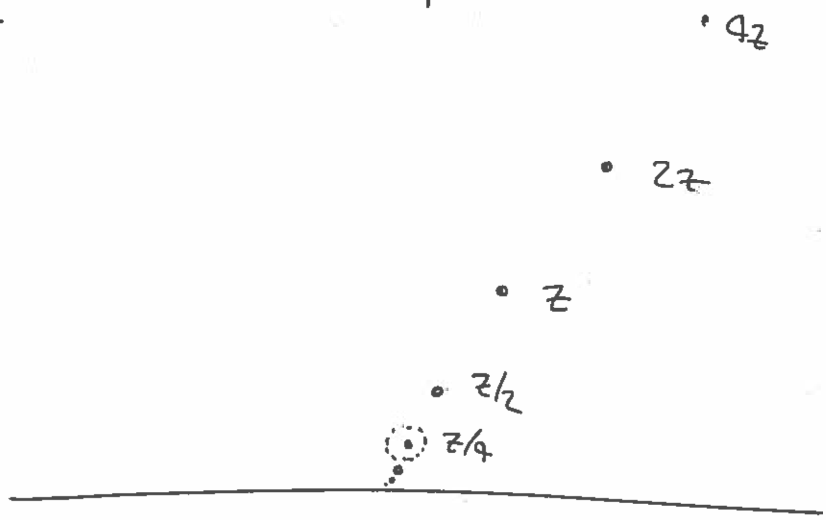
<u>Prop</u> Let Γ be a subgroup of $\text{Möb}(\mathbb{H})$ or $\text{Möb}(\mathbb{D})$.

The following are equivalent:

(1) Γ is a Fuchsian group

(2) $\forall z \in \mathbb{H}$ (or $\mathbb{D}$) the orbit $\Gamma(z) \subset \mathbb{H}$ (or $\mathbb{D}$) is discrete.

<u>Example</u> Let $\Gamma = \{\gamma_n(z) = 2^n z,\ n \in \mathbb{Z}\}$.



$\Gamma(z) = \{2^n z,\ n \in \mathbb{Z}\}$ is discrete $\forall z \in \mathbb{H}$.

Hence Γ is a Fuchsian group.

Jan 2015

SECTION B

Answer **TWO** of the three questions

B5. Consider the following statements. In each case, state whether the statement is true or false and justify your answer by giving either a proof or a counterexample. You will not be awarded any marks for guessing true or false without attempting to justify your answer.

(i) Let $\gamma_1, \gamma_2 \in \text{Möb}(\mathbb{H})$ be two Möbius transformations of $\mathbb{H}$. Then $\gamma_1\gamma_2$ is a Möbius transformation.

[4 marks]

(ii) Let $\tau(\gamma)$ denote the trace of the Möbius transformation γ of $\mathbb{H}$. Then, for every $\gamma_1, \gamma_2 \in \text{Möb}(\mathbb{H})$, we have $\tau(\gamma_1\gamma_2) = \tau(\gamma_1)\tau(\gamma_2)$.

[2 marks]

(iii) Conjugacy between Möbius transformations of $\mathbb{H}$ is an equivalence relation.

[6 marks]

(iv) Let $\gamma_1(z) = z + 1$, $\gamma_2(z) = z - 1$. Then γ_1 and γ_2 are conjugate Möbius transformations of $\mathbb{H}$.

[4 marks]

(v) There exist parabolic Möbius transformations $\gamma_1, \gamma_2 \in \text{Möb}(\mathbb{H})$ such that $\gamma_1\gamma_2$ is hyperbolic.

[4 marks]

(vi) Let L denote the geodesic in $\mathbb{H}$ with endpoints at -2 and 2. There exists a Möbius transformation $\gamma \in \text{Möb}(\mathbb{H})$ that maps L to itself but interchanges the endpoints.

[6 marks]

(vii) There exists a Möbius transformation $\gamma \in \text{Möb}(\mathbb{H})$ that maps the hyperbolic triangle Δ_1 with vertices at $\infty, i, 1$ to the hyperbolic triangle Δ_2 with vertices at $\infty, i, (-1 + i\sqrt{3})/2$.

[4 marks]

B5 (i) This is true. The composition of two Möbius transformations is a Möbius transformation. See Exercise 3.4. (And remember that you need to check that $\gamma_1\gamma_2$ is a Möbius transformation by checking that '$ad - bc = 1$'.)

Some people queried whether $\gamma_1\gamma_2$ meant the composition $\gamma_1 \circ \gamma_2$ or the product $\gamma_1(z)\gamma_2(z)$. As I said many times in the course, we only ever compose Möbius transformations together (and you should recoil in horror at the thought of multiplying them). However if, in your answer, you wrote or made clear that you were interpreting $\gamma_1\gamma_2$ as the product and gave a reasoned answer as to why it wasn't always a Möbius transformation, then I gave you full credit.

(ii) This is false. Some of you constructed highly elaborate counter-examples. The simplest is to take $\gamma_1(z) = \gamma_2(z) = z$, the identity transformation. Then $\tau(\gamma_1) = \tau(\gamma_2) = \tau(\gamma_1\gamma_2) = 4 \neq \tau(\gamma_1)\tau(\gamma_2)$.

A very large number of you took arbitrary transformations $\gamma_1(z) = (a_1z + b_1)/(c_1z+d_1)$, $\gamma_2(z) = (a_2z+b_2)/(c_2z+d_2)$ (in normalised form, I hope), worked out the composition $\gamma_1\gamma_2$, calculated the traces of $\gamma_1, \gamma_2, \gamma_1\gamma_2$ and then boldly stated that it was clear that these were different in general. This isn't a proof! If you are asked to find a counterexample then you actually need to find one (and finding just one will do); you aren't asked for a method which, in principle and with a bit of work, will produce a large number of counterexamples.

(iii) This is true. I think everybody who attempted this got this right.

(iv) This is false. Stating that a parabolic Möbius transformation is conjugate either to $z \mapsto z+1$ or to $z \mapsto z-1$ isn't sufficient: this doesn't say that $z \mapsto z+1, z \mapsto z-1$ aren't conjugate.

You can't say that the conjugacy must be of the form $\gamma(z) = kz$ and then deduce a contradiction—you have to show that $g\gamma_1 \neq \gamma_2 g$ for any $g \in \text{Möb}(\mathbb{H})$.

See the solution to Exercise 10.3 for how to do this.

(v) This is true. Take $\gamma_1(z) = z + 1$. This has matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. There's no point taking γ_2 to also be a translation, as the composition of two translations is also a translation and so parabolic. Instead, think what's the next simplest example of a parabolic transformation. It has to have trace 4, so the Möbius transformation with matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ might be worth looking at. Take $\gamma_2(z) = z/(z+1)$. Then $\gamma_1\gamma_2(z) = (2z + 1)/(z + 1)$, which is normalised and has trace 9, and so is hyperbolic.

You can't say that γ_1 is parabolic and so conjugate to a translation, γ_2 is parabolic and so conjugate to a translation, hence we can assume both γ_1 and γ_2 are translations, hence their composition is a translation and so parabolic. It is correct to say that a parabolic transformation γ_1 is conjugate to a translation; however, this conjugacy depends on the parabolic transformation (it's a change of coordinates that maps the fixed point of γ_1 to ∞). There's no reason why the same conjugacy is going to work simultaneously for both γ_1, γ_2 if they have different fixed points (indeed, this is what makes the example above work).

(vi) This is true. There are two slog-it-out methods and a quick method to see this. One slog-it-out method is to suppose that $\gamma(z) = (az+b)/(cz+d)$ maps -2 to 2 and 2 to -2, use this to deduce two relationships between a, b, c, d, and then find (by trial and error) suitable values of a, b, c, d which satisfy these relationships and the fact that $ad - bc = 1$.

Another slog-it-out method is to take find a Möbius transformation γ that maps the geodesic between -2 and 2 to the imaginary axis, compose this with the map $z \mapsto -1/z$ to interchange the endpoints 0 and ∞, and then map the imaginary axis back to the geodesic from -2 to 2. Many of those who tried this got confused as to whether one of your maps was mapping to or from the geodesic between -2 and 2 and the imaginary axis.

The quick method is just to note that $\gamma(z) = -4/z$ is a Möbius transformation with the required properties.

(vii) This is false. Either you can note that the two triangles have different areas (by the Gauss-Bonnet Theorem) and so—as Möbius transformations preserve area—there cannot be a Möbius transformation that maps Δ_1 to Δ_2. Alternatively, just note that if such a Möbius transformation existed then it would have to map vertices to vertices, but Δ_1 has two vertices on the boundary whereas Δ_2 has one vertex on the boundary. As Möbius transformations map the boundary to itself, this is impossible.