

What did we do last time?

- The Gauss-Bonnet Theorem

$$\text{Area}_{\text{H}}(\text{hyp triangle}) = \pi - (\alpha + \beta + \gamma)$$

$$\text{Area}_{\text{H}}(\text{hyp } n\text{-gon}) = (n-2)\pi - \sum_{j=1}^n \alpha_j.$$

- \exists a tiling of the hyp plane by regular hyp. n -gons with k polygons meeting at each vertex

$$\Leftrightarrow \frac{1}{n} + \frac{1}{k} < \frac{1}{2} \quad (\text{so infinitely many tilings!})$$

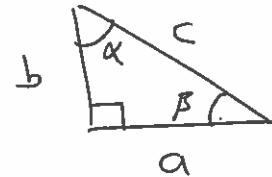
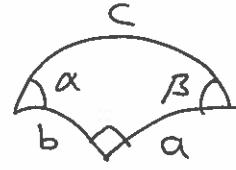
What will we do today?

- Hyperbolic trig!
- Start to classify Möbius txs.

①

8. Hyperbolic triangles & trig

Right-angled triangles:



Proposition

$$(1) \sin \alpha = \frac{\sinh a}{\sinh c}$$

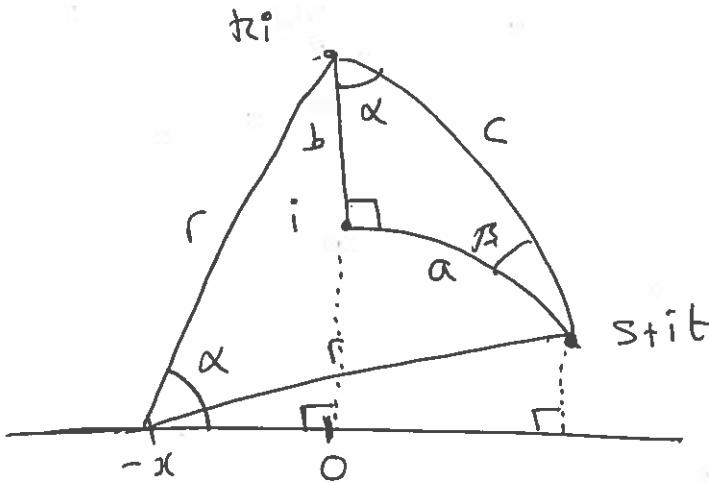
$$(2) \cos \alpha = \frac{\tanh b}{\tanh c}$$

$$(3) \tan \alpha = \frac{\tanh a}{\sinh b}$$

Pf We prove (3) - the others follow by trig/hyp trig algebra.

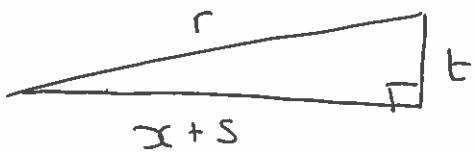
$$\text{Recall: } \cosh d_H(z, w) = 1 + \frac{|z-w|^2}{2 \operatorname{Im} z \operatorname{Im} w}$$

As Möbius tx are conformal, we can move the right angle to i & the side b to the imag. axis without changing the angles.

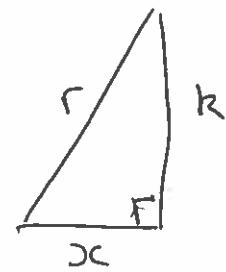


$s^2 + t^2 = 1$ as it lies on the unit circle.

$zi, stit$ lie on a semi-circle with some centre $(at -x)$ and some radius r .



$$\begin{aligned}(x+s)^2 + t^2 &= r^2 \\ x^2 + 2xs + s^2 + t^2 &= r^2 \\ x^2 + 2xs + 1 &= r^2\end{aligned}$$



$$x^2 + k^2 = r^2$$

Hence $x^2 + k^2 = x^2 + 2xs + 1$

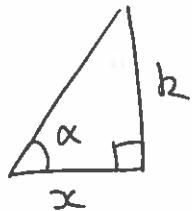
$$\Rightarrow x = \frac{k^2 - 1}{2s}$$

$$\cosh a = \cosh d_H(\text{st}t, i) = 1 + \frac{|s+i(t-1)|^2}{2t} = \frac{1}{t}$$

$$\Rightarrow \tanh a = s.$$

$$\cosh b = \cosh d_H(ki, i) = \dots = \frac{k^2 + 1}{2k}.$$

$$\Rightarrow \sinh b = \frac{k^2 - 1}{2k}.$$

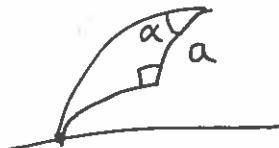


$$\tan \alpha = \frac{k}{x} = \frac{2sk}{k^2 - 1} = \frac{\tanh a}{\sinh b}$$

D.

Angle of parallelism

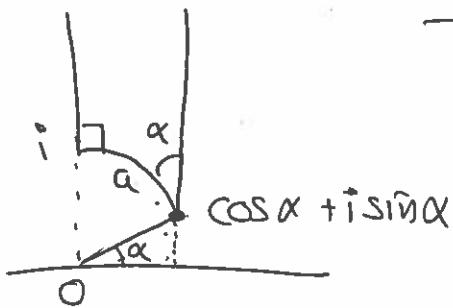
Let Δ be a right-angled triangle with one ideal vertex and side of finite hyperbolic length a . Let α be the remaining angle. Then



$$(1) \sin \alpha = \frac{1}{\cosh a} \quad (2) \cos \alpha = \frac{1}{\coth a}$$

$$(3) \tan \alpha = \frac{1}{\sinh a}.$$

Pf Apply a Möbius tx so that the triangle looks like:



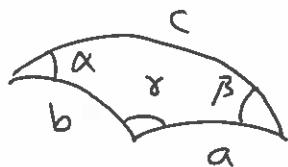
The side of length a lies along the ~~unit circle~~ unit circle.

(3)

$$\begin{aligned}
 \cosh a &= \cosh d_{\mathbb{H}}(\cos \alpha + i \sin \alpha, i) \\
 &= 1 + \frac{|\cos \alpha + i(\sin \alpha - 1)|^2}{2 \sin \alpha} \\
 &= 1 + \frac{\cos^2 \alpha + (\sin \alpha - 1)^2}{2 \sin \alpha} = \dots = \frac{1}{\sin \alpha}
 \end{aligned}$$

□

Non-right angled triangles



$$\text{Hyp sine rule: } \frac{\sin \alpha}{\sinh a} = \frac{\sin \beta}{\sinh b} = \frac{\sin \gamma}{\sinh c}.$$

Cosine rule I

$$\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos \gamma$$

Cosine rule II

$$\cosh c = \frac{\cos \alpha \cos \beta + \cos \gamma}{\sin \alpha \sin \beta}$$

9. Fixed points of Möbius txs

We ~~classify~~ want to classify Möbius txs.

$$\text{Let } \gamma(z) = \frac{az+b}{cz+d} \quad a, b, c, d \in \mathbb{R} \quad ad - bc > 0.$$

Defn Let $z_0 \in \mathbb{H} \cup \partial \mathbb{H}$ be a fixed point of γ if
 $\gamma(z_0) = z_0$.

Rmk If $\gamma = \text{id}$ then γ fixes every point in $\mathbb{H} \cup \partial \mathbb{H}$.

Assume $\gamma \neq \text{id}$.

④

Q: When is ∞ a fixed point?

We defined $\gamma(\infty) = \frac{a}{c}$. So $\gamma(\infty) = \infty \iff c=0$.

Q: If ∞ is a fixed point, are there any others?

Let $\gamma(z) = \frac{az+b}{d}$. Then $\gamma(z_0) = z_0 \iff \frac{az_0+b}{d} = z_0$
 $\iff z_0 = \frac{b}{d-a}$.

If $a=d$ then we get ∞ again

If $a \neq d$ then we get another fixed pt on ∂H .

Q: What if ∞ is not fixed?

So $c \neq 0$. Then $\gamma(z_0) = z_0 \iff \frac{az_0+b}{cz_0+d} = z_0$
 $\iff cz_0^2 + (d-a)z_0 - b = 0$

Hence there are either

2 real solutions - ie two fixed pts on ∂H , none in H .
(γ is hyperbolic)

1 real solution - ie one fixed pt on ∂H , none in H .
(γ is parabolic)

2 complex conjugate - ie no fixed pts on ∂H , ~~one~~ one fixed
schuhans
(γ is elliptic)

What did we do last time?

- Hyperbolic trig: hyperbolic SOH CAH TOA
angle of parallelism
- Started to classify Möbius txs:
Let $\gamma \in \text{Möb}(\mathbb{H})$ (or $\text{Möb}(\mathbb{D})$), $\gamma \neq \text{id}$. Then γ has either

2 fixed pts on $\partial\mathbb{H}$, none in \mathbb{H} 1 fixed pt on $\partial\mathbb{H}$, none in \mathbb{H} no fixed pts on $\partial\mathbb{H}$, 1 in \mathbb{H}	- hyperbolic - parabolic - elliptic
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Cor. Suppose $\gamma \in \text{M\"ob}(H)$, ~~fixed~~. Suppose γ fixes 3 ~~or~~ or more points in $H \cup \partial H$. Then γ is the identity

(1)

M\"obius Let $\gamma_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1}$, $\gamma_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2}$

$$a_i, b_i, c_i, d_i \in \mathbb{R} \quad a_i d_i - b_i c_i > 0 \quad i=1, 2.$$

Then (see one of the exercises/tutorials) $\gamma_1 \gamma_2(z)$ is a M\"obius tx of H with coefficients given by the matrix product $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, let $\gamma_A(z) = \frac{az+b}{cz+d}$.

Note that $\gamma(z) = \frac{\lambda a z + \lambda b}{\lambda c z + \lambda d} = \frac{az+b}{cz+d} \quad \lambda \neq 0$

So $\gamma_{\lambda A} = \gamma_A$ (if $\lambda \neq 0$)

If we take $\lambda = \frac{1}{\sqrt{ad-bc}}$ then we can assume wlog

that $ad-bc = 1$ (ie $\det A = 1$).

Defn We say γ is in normalised form (equivalently γ is normalised) if $\gamma(z) = \frac{az+b}{cz+d}$, $ad-bc = 1$.

Example $\gamma(z) = \frac{10z+1}{z+1}$

/

not in normalised form

$= \frac{\frac{10}{3}z + \frac{1}{3}}{\frac{z}{3} + \frac{1}{3}}$

is in normalised form.

Note: $\gamma(z) = \frac{az+b}{cz+d} = \frac{-az-b}{-cz-d}$ $ad-bc=1$. (2)

both in normalised form.

(ie $\gamma_A(z) = \gamma_{-A}(z)$, $\det A = \det (-A) = 1$)

Aside $SL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$
 $ad-bc=1$

$Möb(\mathbb{H}) \cong SL(2, \mathbb{R}) / \{\pm 1d\} = PSL(2, \mathbb{R})$

10 Classifying Möbius txs

Defn Let $\gamma_1, \gamma_2 \in Möb(\mathbb{H})$. We say γ_1, γ_2 are conjugate if $\exists g \in Möb(\mathbb{H})$ st. $\gamma_1 = g^{-1}\gamma_2 g$.

Rmk (1) g is a "change of co-ordinates".

(2) conjugacy is an equivalence relation.

(3) Suppose $\gamma_1 = g^{-1}\gamma_2 g$. Then

$$\gamma_1(z_0) = z_0 \iff g^{-1}\gamma_2 g(z_0) = z_0$$

$$\iff \gamma_2(g/z_0) = g/z_0$$

$$\iff g(z_0) \text{ is a fixed pt of } \gamma_2.$$

Hence if γ_1 is $\begin{cases} \text{hyp} \\ \text{parab} \\ \text{ellip} \end{cases}$ & γ_2 is $\begin{cases} \text{hyp} \\ \text{parab} \\ \text{conjugate to } \gamma_1 \end{cases} \Rightarrow \gamma_2 \text{ is } \begin{cases} \text{hyp} \\ \text{parab} \\ \text{ellip.} \end{cases}$

Defn Suppose $\gamma(z) = \frac{az+b}{cz+d}$, $ad-bc=1$ is normalised.

$$\text{We define } \tau(\gamma) = (a+d)^2 = (\text{trace } A)^2 = (\text{trace } -A)^2$$

We call $\tau(\gamma)$ the trace of γ .

Suppose ∞ is not a fixed pt (equiv $c \neq 0$). (3)

Then z_0 is a fixed pt for $\gamma(z) = \frac{az+b}{cz+d}$ $ad-bc=1$.

$$\Leftrightarrow \frac{az_0+b}{cz_0+d} = z_0 \Leftrightarrow cz_0^2 + (d-a)z_0 - b = 0$$

$$\Leftrightarrow z_0 = \frac{a-d \pm \sqrt{(a-d)^2 + 4bc}}{2c}$$

Note: $ad-bc=1$, we have $bc = ad-1$. Hence

$$\begin{aligned}(a-d)^2 + 4bc &= a^2 - 2ad + d^2 + 4(ad-1) \\&= a^2 + 2ad + d^2 - 4 = (a+d)^2 - 4 \\&= \tau(\gamma) - 4.\end{aligned}$$

This proves:

- γ is hyperbolic $\Leftrightarrow \tau(\gamma) > 4$
- γ is parabolic $\Leftrightarrow \tau(\gamma) = 4$
- γ is elliptic $\Leftrightarrow \tau(\gamma) \in [0, 4)$

Rmk The above classification is still true if ∞ is a fixed pt

Rmk The above classification also works in \mathbb{D} :

$$\gamma(z) = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}} \quad \alpha, \beta \in \mathbb{C} \quad \text{wlog } |\alpha|^2 - |\beta|^2 = 1 \quad (\text{ie normalised}).$$

$$\text{with } \tau(\gamma) = (\alpha + \bar{\alpha})^2.$$

Parabolic trans

(4)

Example: a translation $\gamma(z) = z + b$ is parabolic
(it has a fixed pt at $\infty \in \partial \mathbb{H}$. - to see

this note $z + b = \frac{1z + b}{0z + 1}$ "c=0")

Exercise Let $\gamma(z) = z + b$. Show that

- if $b > 0$ then γ is conjugate to $z \mapsto z + 1$.
- if $b < 0$ then γ is conjugate to $z \mapsto z - 1$.

Also show: $z \mapsto z + 1$, $z \mapsto z - 1$ are not conjugate.

Proposition Let $\gamma \in \text{M\"ob}(\mathbb{H})$, $\gamma \neq \text{id}$. Then the following are equivalent:

- (1) γ is parabolic (one fixed pt on $\partial \mathbb{H}$, none in \mathbb{H})
- (2) $\tau(\gamma) = 4$
- (3) γ is conjugate to a translation
- (4) γ is conjugate to either $z \mapsto z + 1$
or $z \mapsto z - 1$.