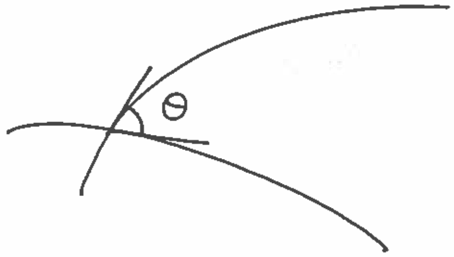
What did we do last time?

- Saw that the geodesics in $\mathbb{H}$ are
  - vertical straight lines
  - semi-circles with real centres
- Saw how to move an arbitrary geodesic & point on it to the imag axis and i, using a Möbius tx

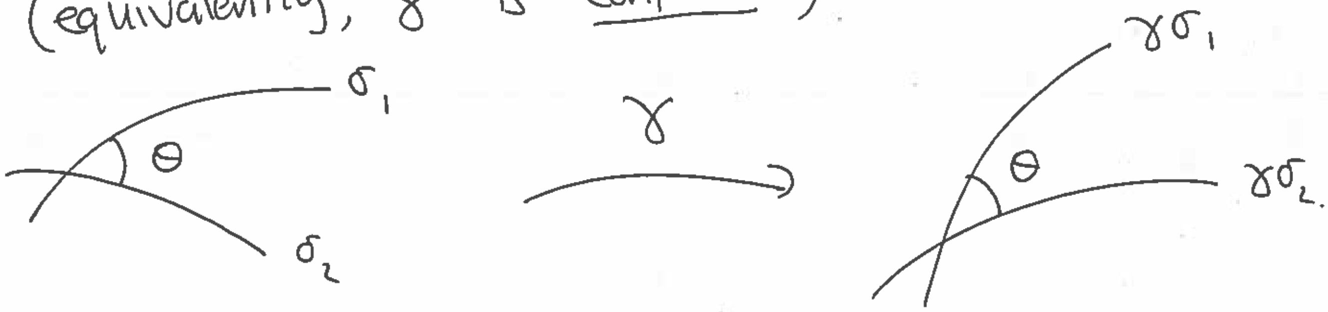
What will we do today?

- Talk (briefly) about hyperbolic area & angles
- Pythagoras' theorem.
- Describe the Poincaré disc model of hyp. geom.

Angles The angle between two geodesics is defined to be the angle between the two tangent vectors at the point of intersection.



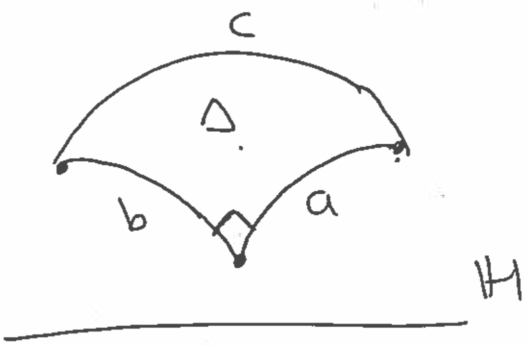
Prop Let $\gamma \in \text{Möb}(\mathbb{H})$. Then γ preserves angles (equivalently, γ is conformal).



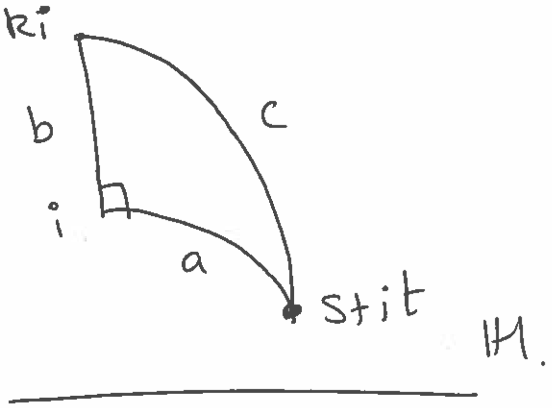
Pythagoras' Theorem

Recall: $\cosh d_{\mathbb{H}}(z,w) = 1 + \dfrac{|z-w|^2}{2\,\mathrm{Im}\,z\,\mathrm{Im}\,w}$.

Prop (Hyp. Pythagoras' Theorem) Let Δ be a hyperbolic triangle with a right angle & with sides of hyperbolic length a, b, c with the side of length c opposite the right-angle. Then $\cosh c = \cosh a \cosh b$.



Pf Apply a Möbius tx so that the right angle is at i & the side of length b is along the imag axis (This is ok ~~&~~ as γ is conformal.)



The side of length a goes through i at right angles to the vertical – ie is horizontal. (2)

So this side must be contained in the unit circle (centre 0, radius r)

Hence $s^2 + t^2 = 1$.

$$\cosh a = \cosh d_{\mathbb{H}}(s+it, i) = 1 + \frac{|s+i(t-1)|^2}{2t}$$

$$= 1 + \frac{s^2+(t-1)^2}{2t} = \frac{1}{t} \quad \text{using } s^2+t^2=1.$$

$$\cosh b = \cosh d_{\mathbb{H}}(i, ki) = 1 + \frac{|(k-1)i|^2}{2k} = \frac{k^2+1}{2k}.$$

$$\cosh c = \cosh d_{\mathbb{H}}(s+it, ki) \overset{\text{easy}}{\underset{\text{algebra}}{=}} \frac{k^2+1}{2kt}$$

$$= \cosh a \cosh b \qquad \square.$$

<u>Hyperbolic area</u> Let $A \subseteq \mathbb{H}$. Then $\text{Area}_{\mathbb{H}} A := \iint_A \frac{dx\,dy}{y^2}$.

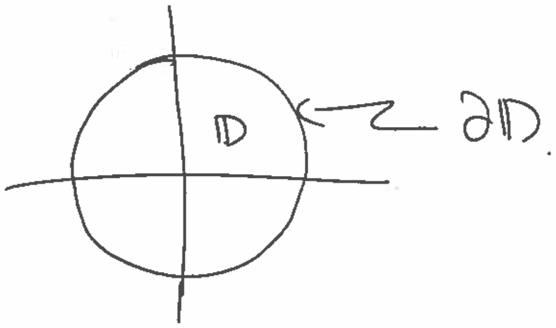
<u>Prop</u> Let $\gamma \in \text{Möb}(\mathbb{H})$. Then γ is <u>area-preserving</u>

ie $\text{Area}_{\mathbb{H}} \gamma(A) = \text{Area}_{\mathbb{H}} A$.

§6 Poincaré disc model

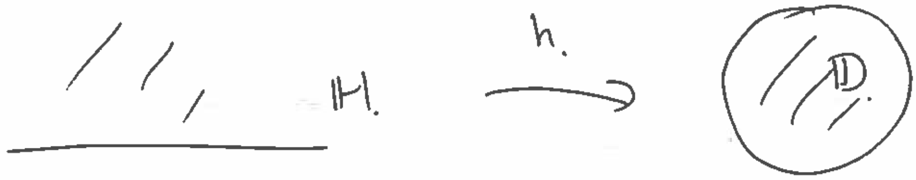
$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ = Poincaré disc

$\partial\mathbb{D} = \{z \in \mathbb{C} \mid |z| = 1\}$ = boundary of $\mathbb{D}$, or circle at infinity.



Define $h : \mathbb{H} \to \mathbb{D}$ $\qquad h(z) = \dfrac{z-i}{iz-1}$.

One can show (exercise!) $h : \mathbb{H} \to \mathbb{D}$, $h : \partial\mathbb{H} \to \partial\mathbb{D}$ are both bijections.



Let $g = h^{-1} : \mathbb{D} \to \mathbb{H}$ $\qquad g(z) = \dfrac{-z+i}{-iz+1}$.

<u>Facts</u>: $g'(z) = \dfrac{-2}{(-iz+1)^2}$, $\quad \operatorname{Im} g(z) = \dfrac{1-|z|^2}{|-iz+1|^2}$.

Let $\sigma : [a,b] \to \mathbb{D}$ be a path in $\mathbb{D}$. How should we define $\text{length}_{\mathbb{D}}(\sigma)$?

Note $g \circ \sigma : [a,b] \to \mathbb{H}$ is a path in $\mathbb{H}$ with hyp length

$$\text{length}_{\mathbb{H}}(g \circ \sigma) = \int_a^b \frac{1}{\operatorname{Im}(g \circ \sigma)(t)} \, |(g \circ \sigma)'(t)| \, dt$$

$$= \int_a^b \frac{1}{\operatorname{Im} g(\sigma(t))} \, |g'(\sigma(t))| \, |\sigma'(t)| \, dt \qquad \text{chain rule.}$$

$$= \int_a^b \frac{\cancel{|-i\sigma(t)+1|^2}}{1-|\sigma(t)|^2} \times \frac{2}{\cancel{|-i\sigma(t)+1|^2}} \times |\sigma'(t)| \, dt.$$

$$= \int_a^b \frac{2}{1-|\sigma(t)|^2} \, |\sigma'(t)| \, dt = \int_\sigma \frac{2}{1-|z|^2}$$

Define $\text{length}_{\mathbb{D}}(\sigma) := \int_\sigma \frac{2}{1-|z|^2}$.

Define $d_{\mathbb{D}}(z,w) := \inf\left\{ \text{length}_{\mathbb{D}}(\sigma) \;\middle|\; \begin{array}{l} \sigma \text{ is a piecewise} \\ \text{diff'ble path} \\ \text{from } z \text{ to } w \end{array} \right\}$.

$= $ a metric on $\mathbb{D}$.

<u>Fact</u> Let $z, w \in \mathbb{H}$. Then $d_{\mathbb{H}}(z,w) = d_{\mathbb{D}}(h(z), h(w))$.

<u>Möbius txs of $\mathbb{D}$</u> Let $\gamma \in \text{Möb}(\mathbb{H})$. Then $h\gamma h^{-1} : \mathbb{D} \to \mathbb{D}$ is an isometry of $\mathbb{D}$ because:

$$d_{\mathbb{D}}(h\gamma h^{-1}(z), h\gamma h^{-1}(w)) = d_{\mathbb{H}}(\gamma(h^{-1}(z)), \gamma(h^{-1}(w))) \quad \text{(by the fact)}$$

$$= d_{\mathbb{H}}(h^{-1}(z), h^{-1}(w)) \quad \text{(as } \gamma \text{ is an isom of } \mathbb{H})$$

$$= d_{\mathbb{D}}(z,w) \quad \text{(by the fact)}$$

Boring algebra shows that $h\gamma h^{-1}$ has the form

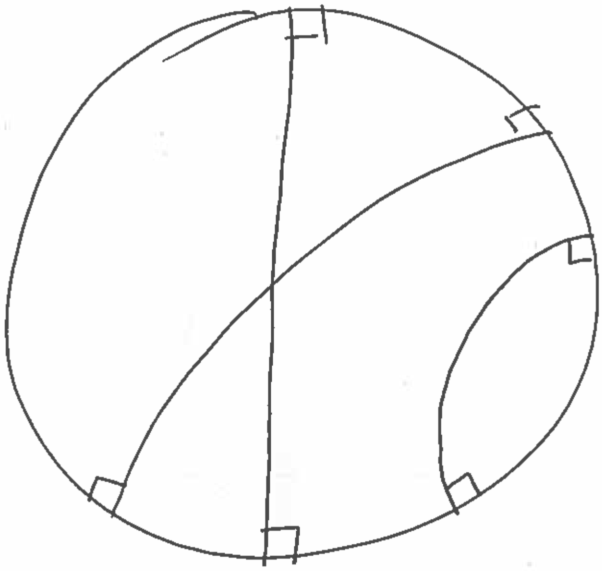
$$\frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}} \qquad \alpha, \beta \in \mathbb{C} \qquad |\alpha|^2 - |\beta|^2 > 0$$

– a Möbius tx of $\mathbb{D}$.

Let $\text{Möb}(\mathbb{D}) = \{\text{Möbius txs of } \mathbb{D}\}$ = a group under composition.

Geodesics in $\mathbb{D}$ The geodesics in $\mathbb{D}$ are

- diameters of $\mathbb{D}$
- arcs of circles that meet $\partial\mathbb{D}$ at right angles.



Favourite geodesic = x-axis

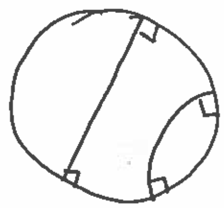
Favourite point = origin.

Area in $\mathbb{D}$ If $A \subseteq \mathbb{D}$ then

$$\mathrm{Area}_{\mathbb{D}}\, A := \iint_A \frac{dz}{(1-|z|^2)^2} = \iint_A \frac{dx\,dy}{(1-(x^2+y^2))^2}$$

What did we do last time?

- Stated that Möbius transformations are
  - conformal (preserve angles)
  - area-preserving.
- Hyperbolic Pythagoras' Theorem: $\cosh c = \cosh a \cosh b$
- Defined the Poincaré disc



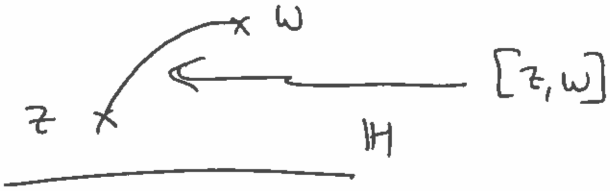
What will we do today?

- The Gauss-Bonnet Theorem
- Study tilings/tessellations of the hyperbolic plane.

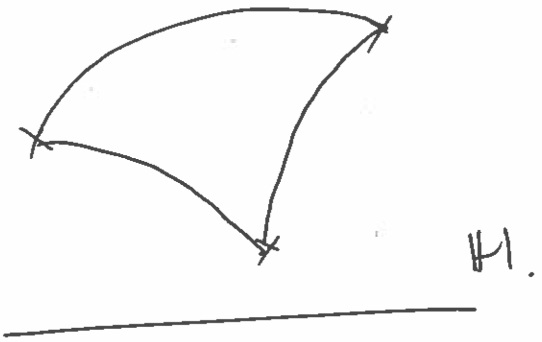
7. Gauss-Bonnet Thm

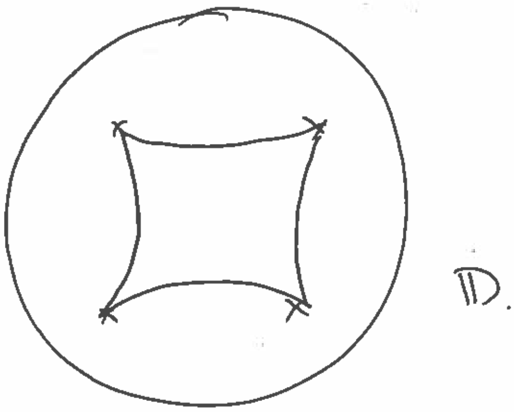
Hyperbolic polygons: let $z_1, \dots, z_n \in \mathbb{H} \cup \partial\mathbb{H}$

Let $[z,w]$ denote the arc of geodesic between z and w

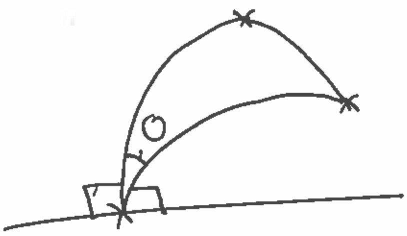


The hyperbolic polygon with vertices $z_1, \dots, z_n$ is the region of $\mathbb{H}$ bounded by $[z_1, z_2], [z_2, z_3], \dots, [z_{n-1}, z_n], [z_n, z_1]$.





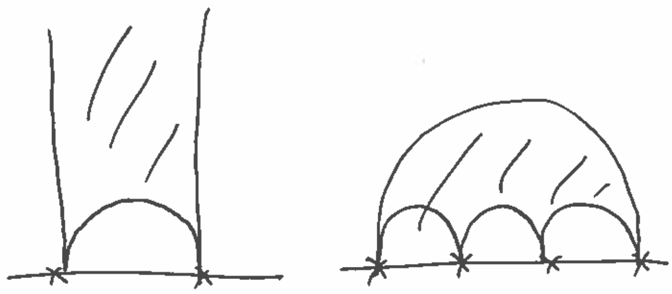
A vertex on the boundary is called an <u>ideal vertex</u>.

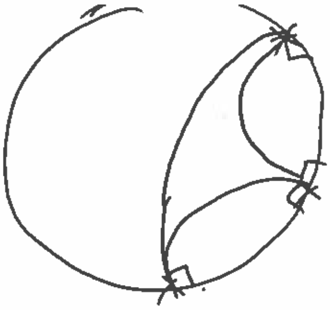


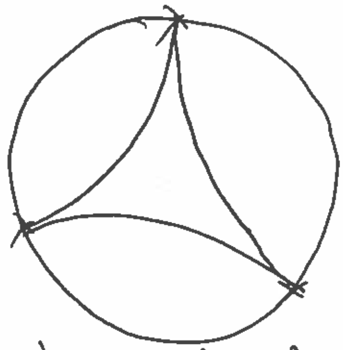
The internal angle at an ideal vertex is 0.

triangle with an ideal vertex at ∞.

If all the vertices are ideal then we call ~~then~~ the polygon an <u>ideal polygon</u>







Thm (Gauss-Bonnet Thm for hyperbolic triangles)
Let Δ be a hyperbolic triangle with internal angles α, β, γ. Then $\text{Area}_{\mathbb{H}} \Delta = \pi - (\alpha + \beta + \gamma)$.

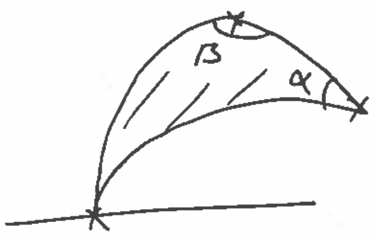
Rmks (1) Unlike Euclid geom, the angles of a triangle determine the area.

(2) $\text{Area}_{\mathbb{H}} \Delta \leq \pi$ with equality iff $\alpha = \beta = \gamma = 0$
ie equality iff Δ is an ideal triangle.

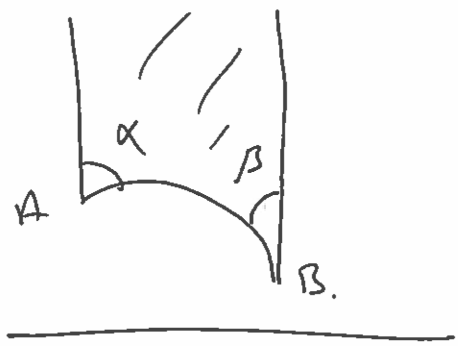
(3) $\sum$ internal angles $< \pi$.

Recall $\text{Area}_{\mathbb{H}} A = \iint_A \dfrac{dx\, dy}{y^2}$.

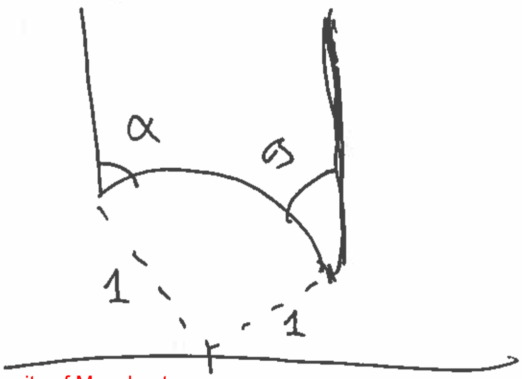
Pf. **Case 1** Δ has at least one ideal vertex.

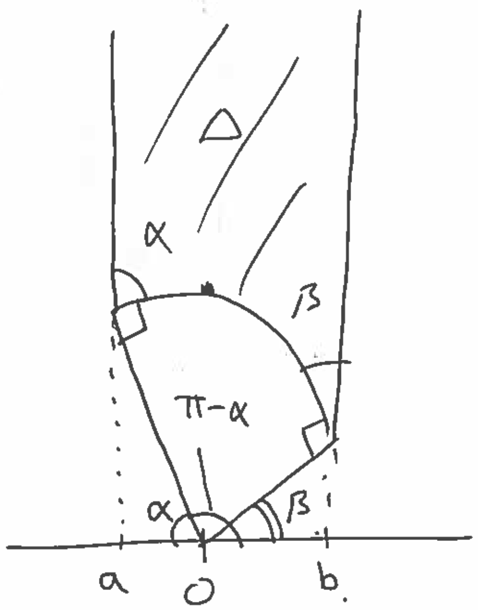


Apply a Möb. tx so that the ideal vertex is at ∞ (does not change area or angles)



apply a translation & dilation so that $[A, B]$ is contained in the unit circle in $\mathbb{C}$ (again, doesn't change area or angles).





$$\text{Area}_{\mathbb{H}}\,\Delta = \iint_{\Delta} \frac{dx\,dy}{y^2}.$$

$$= \int_{x=a}^{b} \int_{y=\sqrt{1-x^2}}^{\infty} \frac{dy}{y^2}\,dx$$

$$= \int_{x=a}^{b} \left(\frac{-1}{y} \Big|_{\sqrt{1-x^2}}^{\infty} \right) dx$$

$$= \int_{x=a}^{b} \frac{dx}{\sqrt{1-x^2}} = \int_{\pi-\alpha}^{\beta} -1\,d\theta$$

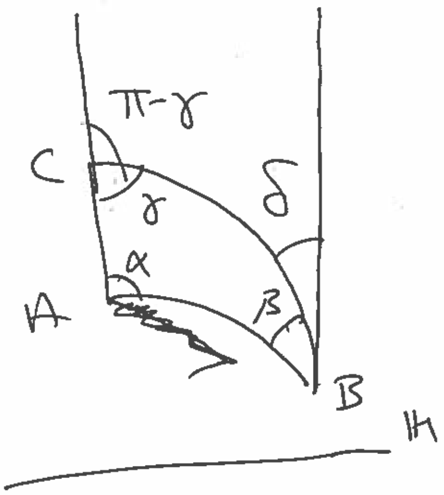
$x = \cos\theta$

x	a	b.
θ	$\pi-\alpha$	β.

$$= \pi - (\alpha+\beta).$$

<u>Case 2.</u> Δ has no vertices on the boundary

For convenience, apply a Möbius tx so that one side is vertical.



Draw in the geodesics from C to ∞ and from B to ∞

$$\text{Area}_{\mathbb{H}}\,ABC = \text{Area}_{\mathbb{H}}\,AB\infty - \text{Area}_{\mathbb{H}}\,BC\infty$$

$$= [\pi - (\alpha + (\beta+\delta))] - [\pi - ((\pi-\gamma)+\delta)]$$

$$= \pi - (\alpha+\beta+\gamma) \qquad \square$$

Gauss-Bonnet for a hyperbolic n-gon

Let P be a hyp. polygon with internal angles $\alpha_1, \ldots, \alpha_n$. Then

$$\text{Area}_{\mathbb{H}} P = (n-2)\pi - \sum_{j=1}^{n} \alpha_j .$$

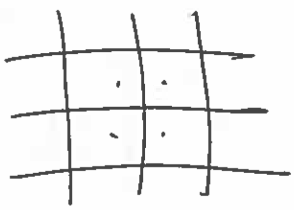
<u>Thm</u> Suppose $\alpha_1, \ldots, \alpha_n$ are s.t. $(n-2)\pi - \sum_{j=1}^{n} \alpha_j > 0$

Then $\exists$ a hyp polygon with internal angles $\alpha_1, \ldots, \alpha_n$.

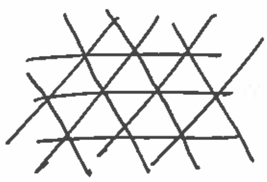
<u>Defn</u> A regular polygon is one where all angles are equal & all sides have the same length.

Q: When can we tile the plane using regular n-gons with k polygons meet at each vertex?

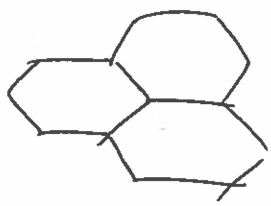
<u>Euclidean</u>



$n=4 \quad k=4$



$n=3 \quad k=6$



$n=6 \quad k=3$.

<u>Prop</u> $\exists$ a tiling of the hyp plane by regular n-gons with k polygons meeting at each vertex $\iff \frac{1}{n} + \frac{1}{k} < \frac{1}{2}$.

<u>Pf</u> $\Leftarrow$ hard.

$\Rightarrow$ Suppose the internal angle is α. As k polygons meet at each vertex $k\alpha = 2\pi$, ie $\alpha = 2\pi/k$

$$\text{Area (poly)} = (n-2)\pi - \sum_{j=1}^{n} \alpha = (n-2)\pi - \frac{2\pi n}{k} > 0$$

Rearrange: $\frac{1}{n} + \frac{1}{k} < \frac{1}{2}$ $\square$