What did we do last time?

- The <u>hyperbolic length</u> of a path $\sigma : [a,b] \to \mathbb{H}$ is
$$\text{length}_{\mathbb{H}}(\sigma) := \int_a^b \frac{1}{\operatorname{Im}\sigma(t)} \, |\sigma'(t)| \, dt$$

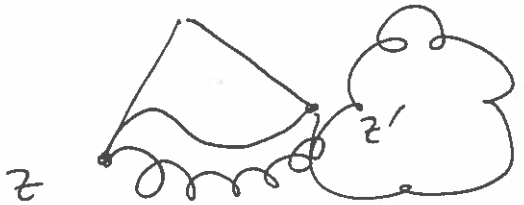
- The <u>hyperbolic distance</u> between two points $z, z' \in \mathbb{H}$ is
$$d_{\mathbb{H}}(z,z') := \inf \left\{ \text{length}_{\mathbb{H}}(\sigma) \;\middle|\; \sigma \text{ is a piecewise diff'ble path in } \mathbb{H} \text{ from } z \text{ to } z' \right\}$$

- $\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0 \quad \alpha, \gamma \in \mathbb{R} \quad \beta \in \mathbb{C}$
  - <u>ALL</u> straight lines & circles in $\mathbb{C}$

- $\alpha z\bar{z} + \beta z + \beta\bar{z} + \gamma = 0 \quad \alpha, \beta, \gamma \in \mathbb{R}$
  - <u>VERTICAL</u> straight lines, circles with <u>REAL</u> centres

What will we do today?

- introduce Möbius transformations
- show Möbius transformations are <u>isometries</u> (they preserve distance).



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Möbius txs

A <u>Möbius tx</u> of $\mathbb{H}$ is a map of the form

$$\gamma(z) = \frac{az+b}{cz+d} \qquad a,b,c,d \in \mathbb{R} \quad ad-bc>0.$$

Let $\text{Möb}(\mathbb{H}) = \{\text{Möbius txs of } \mathbb{H}\}$

<u>Exercise</u> Let $\gamma \in \text{Möb}(\mathbb{H})$. Show γ maps $\mathbb{H}$ to itself bijectively.

<u>Prop</u> $\text{Möb}(\mathbb{H})$ is a group under composition.

The group operation is composition: $\gamma_1\gamma_2(z) := \gamma_1(\gamma_2(z))$.

[We ;<u>NEVER</u>; multiply Mobius txs together

$\gamma_1\gamma_2(z)$ <u>DOES</u> <u>NOT</u> <u>MEAN</u> $\gamma_1(z)\gamma_2(z)$!]

<u>Examples</u> Translations $\gamma(z) = z+b \quad b \in \mathbb{R}$.

$$\left(= \frac{1z+b}{0z+1}\right)$$

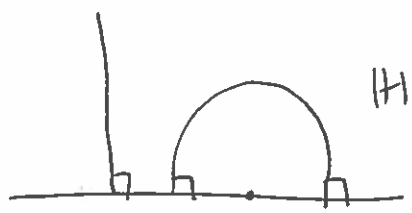
Dilations: $\gamma(z) = kz \quad k>0$.

$$= \left(\frac{kz+0}{0z+1}\right).$$

Inversion in a circle $\gamma(z) = \dfrac{-1}{z}$.

<u>Defn</u> Let $\mathcal{H} = \{$vertical straight lines in $\mathbb{H}$, semi-circles in $\mathbb{H}$ with real centres$\}$

<u>Rmk</u> Sometimes we say "semi-circles that meet $\partial\mathbb{H}$ orthogonally."



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Prop Let $H \in \mathcal{H}$. Let $\gamma \in \text{Möb}(\mathbb{H})$. Then $\gamma(H) \in \mathcal{H}$.

Pf Let $\gamma(z) = \dfrac{az+b}{cz+d}$ $\quad a,b,c,d \in \mathbb{R}$ $\quad ad-bc>0$.

Let H have eqn $\alpha z\bar{z} + \beta z + \beta \bar{z} + \gamma = 0$ $\quad \alpha, \beta, \gamma \in \mathbb{R}$.

Let $w = \gamma(z)$ where $z \in H$. We want to show w satisfies an eqn of the form $\alpha' z\bar{z} + \beta' z + \beta' \bar{z} + \gamma' = 0$ $\alpha', \beta', \gamma' \in \mathbb{R}$.

As $w = \gamma(z) = \dfrac{az+b}{cz+d}$, we have $z = \dfrac{dw-b}{-cw+a}$. Hence

$$\alpha\left(\frac{dw-b}{-cw+a}\right)\left(\frac{d\bar{w}-b}{-c\bar{w}+a}\right) + \beta\left(\frac{dw-b}{-cw+a}\right) + \beta\left(\frac{d\bar{w}-b}{-c\bar{w}+a}\right) + \gamma = 0$$

$$\alpha(dw-b)(d\bar{w}-b) + \beta(dw-b)(-c\bar{w}+a) + \beta(d\bar{w}-b)(-cw+a) + \gamma(-cw+a)(-c\bar{w}+a) = 0$$

$$\alpha' w\bar{w} + \beta' w + \beta' \bar{w} + \gamma' = 0$$

where
$$\alpha' = \alpha d^2 - 2\beta cd + \gamma c^2$$
$$\beta' = -\alpha bd + \beta ad + \beta bc - \gamma ac$$
$$\gamma' = \alpha b^2 - 2\beta ab + \gamma a^2$$

– the eqn of either a vertical straight line in $\mathbb{C}$ or a circle with a real centre in $\mathbb{C}$.

As γ maps $\mathbb{H}$ to itself, this shows that $\gamma(H) \in \mathcal{H}$ $\square$

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§4 Möbius txs of $\mathbb{H}$ & geodesics in $\mathbb{H}$

Extend the defn of Möbius txs to the boundary $\partial\mathbb{H}$.

Let $\gamma(z) = \dfrac{az+b}{cz+d}$. $\quad a, b, c, d \in \mathbb{R}$.

Then γ maps the real axis to the real axis, except when $z = -\frac{d}{c}$. Define $\gamma\left(-\frac{d}{c}\right) = \infty \in \partial\mathbb{H}$.

How do we define $\gamma(\infty) = ?$

$$\gamma(z) = \frac{az+b}{cz+d} = \frac{a + b/z}{c + d/z} \to \frac{a}{c} \quad \text{as } z \to \infty.$$

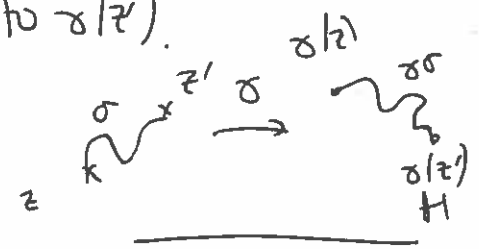
Define $\gamma(\infty) = a/c$.

<u>Exercise</u> Show that if $a, b, c, d \in \mathbb{R}$ $ad - bc \neq 0$ then γ maps $\partial\mathbb{H}$ to $\partial\mathbb{H}$ bijectively.

<u>Prop</u> Let $\gamma \in \text{Möb}(\mathbb{H})$, $z, z' \in \mathbb{H}$. Then

$$d_{\mathbb{H}}(\gamma(z), \gamma(z')) = d_{\mathbb{H}}(z, z')$$

<u>Pf</u> σ is a path from z to z' $\iff$ $\gamma \circ \sigma$ is a path from $\gamma(z)$ to $\gamma(z')$.



It is sufficient to prove that $\text{length}_{\mathbb{H}}(\gamma \circ \sigma) = \text{length}_{\mathbb{H}}(\sigma)$.

Let $\gamma(z) = \dfrac{az+b}{cz+d}$ $\quad a, b, c, d \in \mathbb{R}$ $\quad ad - bc > 0$.

<u>Exercise</u> $\quad |\gamma'(z)| = \dfrac{ad-bc}{|cz+d|^2}$, $\quad \text{Im}\,\gamma(z) = \dfrac{ad-bc}{|cz+d|^2} \times \text{Im}(z)$

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Let $\sigma : [a,b] \to \mathbb{H}$. Then.

$$\text{length}_{\mathbb{H}}(\gamma \circ \sigma) = \int_a^b \frac{1}{\operatorname{Im} \gamma \circ \sigma(t)} \, |(\gamma \circ \sigma)'(t)| \, dt$$

$$= \int_a^b \frac{1}{\operatorname{Im} \gamma(\sigma(t))} \, |\gamma'(\sigma(t))| \, |\sigma'(t)| \, dt. \quad \left(\text{chain rule}\right)$$

$$= \int_a^b \frac{\cancel{|c\sigma(t)+d|^2}}{\cancel{ad-bc}} \times \frac{1}{\operatorname{Im} \sigma(t)} \times \frac{\cancel{ad-bc}}{\cancel{|c\sigma(t)+d|^2}} \times |\sigma'(t)| \, dt$$

$$= \int_a^b \frac{1}{\operatorname{Im} \sigma(t)} \, |\sigma'(t)| \, dt = \text{length}_{\mathbb{H}}(\sigma) \qquad \square$$

We have shown that Möbius trs of $\mathbb{H}$ are <u>isometries</u> (= distance preserving transformations) of $\mathbb{H}$.

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What did we do last time?

- Defined Möbius txs of $\mathbb{H}$: $\gamma(z) = \dfrac{az+b}{cz+d}$ $\quad a,b,c,d \in \mathbb{R}$, $ad-bc>0$

- Proved: if H is a vert. straight line or semi-circle with real centre in $\mathbb{H}$ then so is $\gamma(H)$

- Proved: Möb txs are <u>isometries</u> of $\mathbb{H}$

What will we do today?

- Find the geodesics (= paths of shortest length) in $\mathbb{H}$

Recall $\mathcal{H} = \{$vertical straight lines, circles with real centres in $\mathbb{H}\}$

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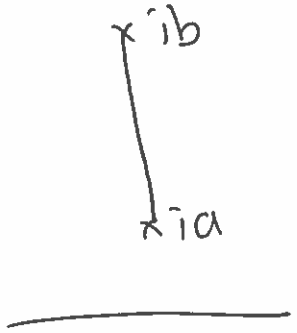
The imag. axis is a geodesic.

<u>Prop</u>. Let $0<a<b$. The hyperbolic distance from ia to ib is $\log b/a$. The arc of ~~geodes~~ imag. axis from ia to ib is the unique path that has hyp. length $\log b/a$ — any other path has strictly larger hyp. length.

<u>Pf</u>. Let $\sigma(t) = it \quad a \le t \le b$.



Then $|\sigma'(t)| = 1$, $\operatorname{Im}\sigma(t) = t$. So

$$\text{length}_{\mathbb{H}}(\sigma) = \int_a^b \frac{dt}{t} = \log t\Big|_a^b$$

$$= \log b/a.$$

Let $\sigma(t) = x(t) + iy(t) \quad 0 \le t \le 1$. ~~Then~~ be a path from ia to ib. Then $y(0) = a \quad y(1) = b$.

$$\text{length}_{\mathbb{H}}(\sigma) = \int_0^1 \frac{1}{\operatorname{Im}\sigma(t)} |\sigma'(t)|\, dt$$

$$= \int_0^1 \frac{1}{y(t)} \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

$$\geq \int_0^1 \frac{1}{y(t)} y'(t)\, dt = \log y(t)\Big|_0^1 = \log b/a.$$

with equality iff $x'(t) = 0$ iff $x(t) = \text{constant}$ iff σ is the arc of geodesic from ia to ib $\square$.

Mapping to the imag axis.

<u>Lemma</u> Let $H \in \mathcal{H}$. Then $\exists\, \gamma \in \text{Möb}(\mathbb{H})$ st $\gamma(H) =$ imag axis

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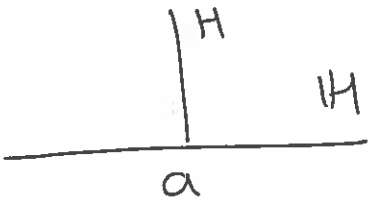
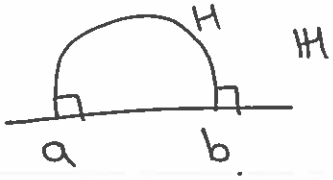
Pf. Case 1 H is the vertical straight line $\mathrm{Re}(z)=a$.



Take $\gamma(z) = z - a \in \text{Möb}(\mathbb{H})$

Then $\gamma(H) =$ imag axis.

Case 2 H is a ½ circle with real centre, meeting $\partial\mathbb{H}$ at $a<b$.



Elements of $\mathcal{H}$ are uniquely determined by their end points in $\partial\mathbb{H}$. The end points of the imag. axis are $0, \infty$.

We want a Möbius tx that maps $\{a, b\}$ to $\{0, \infty\}$

$$\cancel{\gamma(z) = \frac{z-a}{z-b}} \qquad \gamma(z) = \frac{z-b}{z-a}.$$

$$\cancel{1\times(-b) - (-a)\times 1 = a-b<0} \qquad 1\times(-a) - (-b)\times 1 = b-a > 0$$

not a Möbius tx.

Take $\gamma(z) = \dfrac{z-b}{z-a}$. $\square$

More generally:

Lemma Let $H \in \mathcal{H}$, $z_0 \in H$. Then $\exists\, \gamma \in \text{Möb}(\mathbb{H})$ s.t. $\gamma(H) =$ imag. axis, $\gamma(z_0) = i$.

Pf. Choose γ_1 as in the previous lemma so that $\gamma_1(H) =$ imag axis. Let $\gamma_1(z_0) = i/k$ for some $k>0$.

Take $\gamma_2(z) = kz \in \text{Möb}(\mathbb{H})$. Then $\gamma_2(\text{imag axis}) =$ imag axis, and $\gamma_2(i/k) = i$.

Take $\gamma = \gamma_2 \circ \gamma_1 \in \text{Möb}(\mathbb{H})$. $\square$

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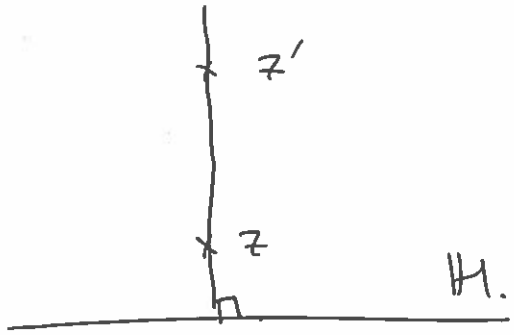
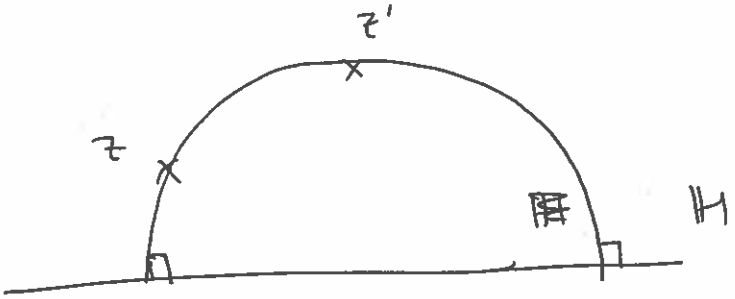
§5 More on the geodesics in $\mathbb{H}$

Theorem. The geodesics in $\mathbb{H}$ are: vertical straight lines in $\mathbb{H}$ or $1/2$-circles that meet $\partial\mathbb{H}$ orthogonally. Moreover, through any two points $z, z' \in \mathbb{H}$, $\exists$ a unique geodesic through them.

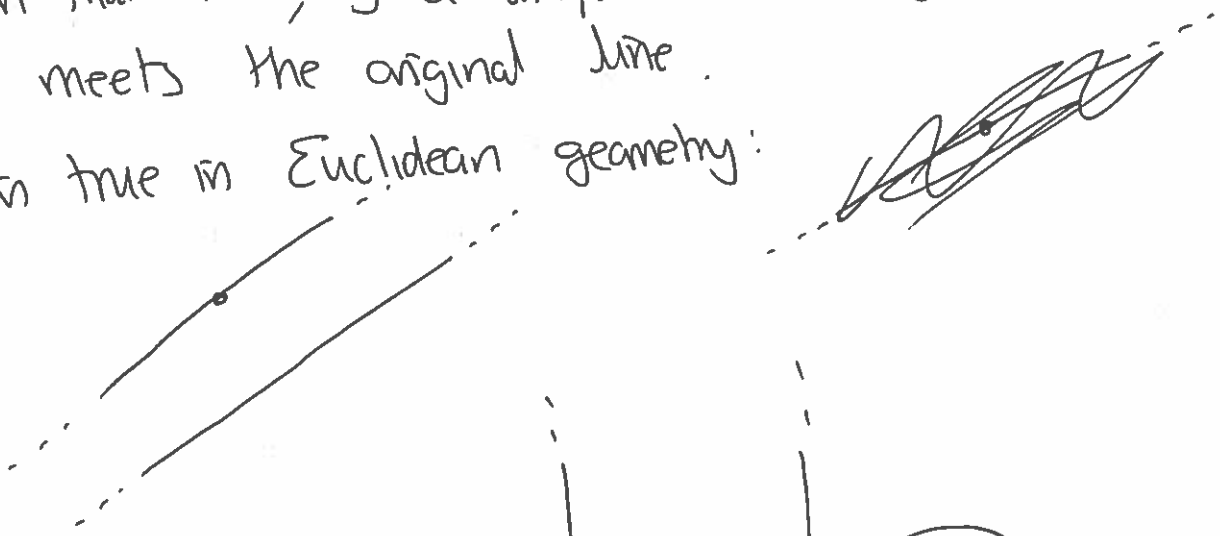




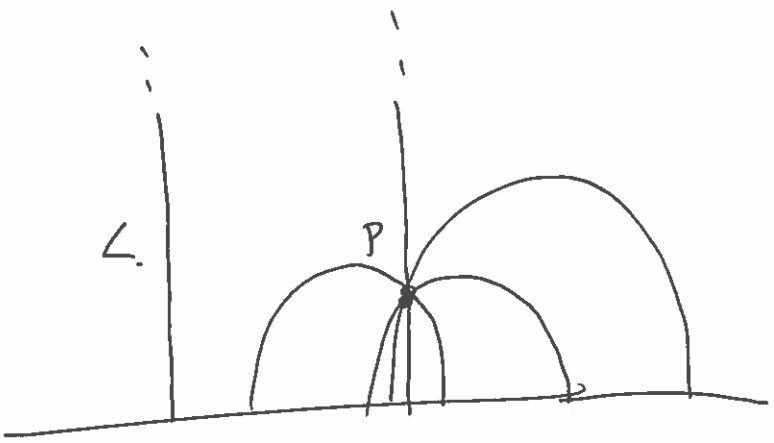
Pf See notes if interested. $\square$

Euclid's parallel postulate says: given a line & a point not on that line, $\exists$ a unique line through the point that never meets the original line.

This is true in Euclidean geometry:



False in hyp geom.



There are infinitely many geodesics through P that never meet L.

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<u>Fact:</u> The distance between arbitrary points in $\mathbb{H}$:

Let $z, w \in \mathbb{H}$. Then

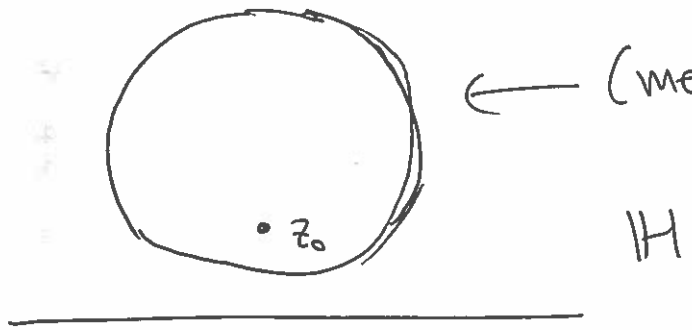
$$\cosh d_{\mathbb{H}}(z,w) = 1 + \frac{|z-w|^2}{2 \operatorname{Im} z \operatorname{Im} w}.$$

<u>Fact</u> <u>Hyperbolic circles.</u>

The hyp. circle with centre $z_0 \in \mathbb{H}$, $r > 0$ is

$$\{ z \in \mathbb{H} \mid d_{\mathbb{H}}(z, z_0) = r \}.$$

Hyperbolic circles are Euclidean circles, but with different centres & different radii.



← (meant to be a circle - sorry!).

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Q1 Let $\gamma_1(z) = \dfrac{a_1 z + b_1}{c_1 z + d_1}$, $\gamma_2(z) = \dfrac{a_2 z + b_2}{c_2 z + d_2}$, $a_i, b_i, c_i, d_i \in \mathbb{R}$, $a_i d_i - b_i c_i > 0$ $(i = 1, 2)$

Does $\gamma_1 \gamma_2$ mean "multiply γ_1, γ_2" or "compose γ_1, γ_2" (delete as appropriate)?

Show that $\gamma_1 \gamma_2$ is a Möbius tx of $\mathbb{H}$.

Q2 Recall that $d_{\mathbb{H}}(ai, bi) = \log b/a$ if $0 < a < b$. Also recall that the imag axis is the unique geodesic through ai, bi. Find the hyperbolic mid-point of the arc of geodesic from $2i$ to $8i$ (ie the point that splits the geodesic between $2i$, $8i$ into two parts, each of equal hyperbolic length).

Q3 Recall that vert. straight lines / semi circles with real centres in $\mathbb{H}$ have equations of the form $\alpha z \bar{z} + \beta z + \beta \bar{z} + \gamma = 0$ $\alpha, \beta, \gamma \in \mathbb{R}$.

Find the equation of the geodesic (ie find α, β, γ) that passes through $-3 + 4i$, $4 + 3i$.

Find a Möbius tx that maps this geodesic to the imag. axis

[Q1 is part of Ex 3.4. Qs 2,3 are parts of past exam questions]

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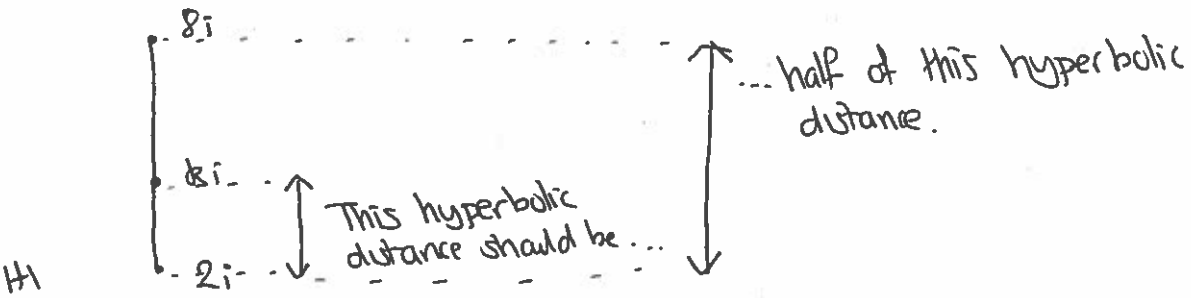
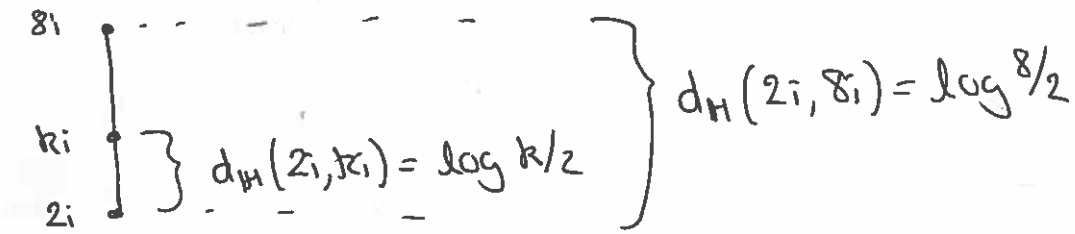
Q1: See solution to Ex 3.4

Q2: Suppose the hyperbolic midpoint occurs at ki:



Recall $d_{\mathbb{H}}(ai, bi) = \log b/a$ if $0 < a < b$.



So $\frac{1}{2} d_{\mathbb{H}}(2i, 8i) = d_{\mathbb{H}}(2i, ki) \iff \frac{1}{2} \log 8/2 = \log k/2.$

$\iff \frac{1}{2} \log 4 = \log k/2.$

$\iff \log 4^{1/2} = \log k/2.$

$\iff \sqrt{4} = k/2 \iff 2 = k/2 \iff k = 4.$

So the hyperbolic midpoint between $2i, 8i$ is at $4i$.

[Popular – but wrong – answers include:

- $5i$ Reasoning: $\frac{2i + 8i}{2} = 5i$.

  Reason why its wrong: this is the Euclidean mid-point.
  Note that $d_{\mathbb{H}}(2i, 5i) = \log 5/2 \neq \frac{1}{2} d_{\mathbb{H}}(2i, 8i) = \log 2$

- $(2 + \log 2)i$ Reasoning: $d_{\mathbb{H}}(2i, 8i) = \log 8/2 = \log 4$. So half the distance is $\frac{1}{2} \log 4 = \log 2$.

  So the midpoint must be at ~~2+log 2~~ $(2 + \log 2)i$

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Reason why this is wrong: the mistake here is to assume that distance in hyp-geom works nicely with the vector space structure of $\mathbb{C}$ (or $\mathbb{R}^2$) – it doesn't.

Note $d_{\mathbb{H}}(2i, (2+\log 2)i) = \log\left(\frac{2+\log 2}{2}\right)$

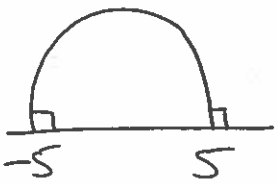
$$\neq \frac{1}{2} d_{\mathbb{H}}(2i, 8i) = \log 2.$$

<u>Q3</u> First note $-3+4i$, $4+3i$ lie on a semi-circle not a vertical straight line. So $\alpha \neq 0$. Dividing through by α allows us to assume without loss of generality that $\alpha = 1$. Both $-3+4i$, $4+3i$ lie on $z\bar{z} + \beta z + \beta \bar{z} + \gamma = 0$, so substituting in gives two equations:

$$\left.\begin{aligned} (-3+4i)(-3-4i) + \beta(-3+4i) + \beta(-3-4i) + \gamma &= 0 \\ (4+3i)(4+3i) + \beta(4+3i) + \beta(4-3i) + \gamma &= 0 \end{aligned}\right]$$

$$\left.\begin{aligned} 25 - 6\beta + \gamma &= 0 \\ 25 + 8\beta + \gamma &= 0 \end{aligned}\right]$$

Hence $\beta = 0$, $\gamma = -25$. So we have $z\bar{z} - 25 = 0$ (or $|z| = 5$ – it's a (Euclidean) circle centre 0 radius 5)



Take $\gamma(z) = \frac{z-5}{z+5}$. This is a Möbius tx as "ad-bc" $= 1 \times 5 - (-5) \times 1 = 10 > 0$.

Note: $\gamma(5) = 0$, $\gamma(-5) = \infty$. So γ maps this geodesic to the imag. axis.

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