MATH 32051 Hyperbolic Geometry

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Office hr: Tue 4–5 ATB 2.241.

Lectures: Thur 4–5 Chem G54

Fri 3–4 ATB G107

Tutorial: Fri 4–5 Simon 2-39. Week 2.

~~Books: J. Anderson, Hyperbolic Geometry~~

S. Katok, Fuchsian Groups

A. Beardon, The Geometry of Discrete Groups

you prob. won't need them!

All lecture notes, exercises, solutions, past exams, etc are on the course webpage.

Coursework test: Wk 6 (see your coursework timetable)

Learning outcomes for MATH32052 Hyperbolic Geometry

- Calculate the hyperbolic distance between and the geodesic through points in the hyperbolic plane
- Compare different models (the upper half-plane model and the Poincaré disc model) of hyperbolic geometry
- Prove results (The Gauss-Bonnet Theorem, angle formulæ for triangles, etc as listed in the syllabus) in hyperbolic trigonometry and use them to calculate angles, side lengths, hyperbolic areas, etc, of hyperbolic triangles and polygons.
- Classify Möbius transformations in terms of their actions on the hyperbolic plane.
- Calculate a fundamental domain and a set of side-pairing transformations for a given Fuchsian group.
- Define a finitely presented group in terms of generators and relations.
- Use Poincaré's Theorem to construct examples of Fuchsian groups and calculate presentations in terms of generators and relations.
- Relate the signature of a Fuchsian group to the algebraic and geometric properties of the Fuchsian group and to the geometry of the corresponding hyperbolic surface.

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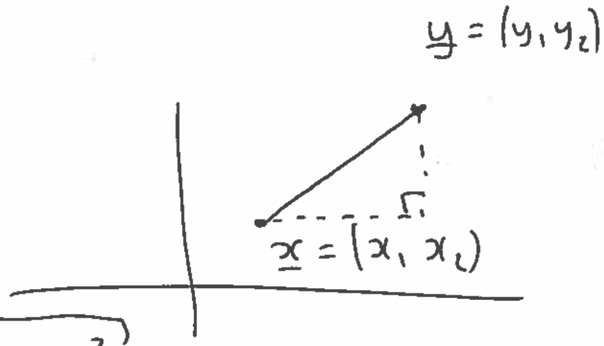
Klein ~~Erlagen~~ Erlangen programme

Given a set X with some structure, a group of transformations (txs) of X that preserves this structure, geometry is the study of ~~a~~ objects that are invariant under these txs.

Eg: Euclidean geometry in $\mathbb{R}^2$

Set $X = \mathbb{R}^2$

Structure = distance in $\mathbb{R}^2$



$$d(\underline{x}, \underline{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

This is a metric (or distance function)

(1) $d(\underline{x}, \underline{y}) \geq 0$ with equality iff $\underline{x} = \underline{y}$.

(2) $d(\underline{x}, \underline{y}) = d(\underline{y}, \underline{x})$

(3) triangle ineq: $d(\underline{x}, \underline{y}) \leq d(\underline{x}, \underline{z}) + d(\underline{z}, \underline{y})$.

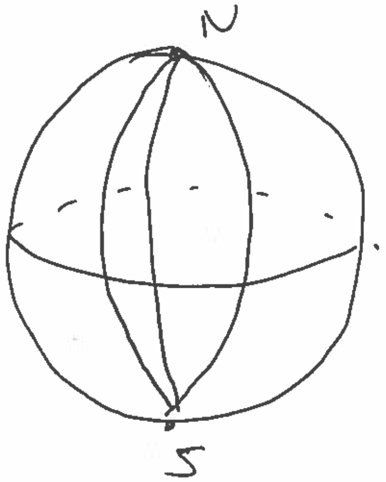
With this metric, rotations, translations, reflections don't change distances (they are isometries). The isometries of $\mathbb{R}^2$ form a group under composition.

Q: Given 2 points in $\mathbb{R}^2$ is there always a straight line between them?

Q: What does "straight" ~~mean~~? mean?

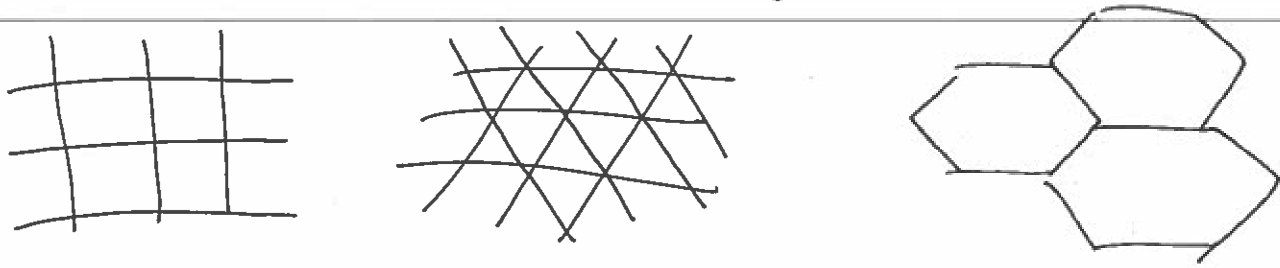
Q: If a tx preserves distances, does it have to preserve area, angles, etc?

In spherical geometry, there are lots of paths of shortest length between ~~the~~ the N and S pole.



We will define hyperbolic geometry in the following way: given two points x, y, look at all paths between them, calculate the hyperbolic length & then take the infimum (minimum). This gives the hyperbolic metric. We then ~~use~~ look for isometries ...

Q: In Euclidean geometry, how many tilings of the plane are there using regular polygons?



— that's it!

In hyp geom there are infinitely many tilings!

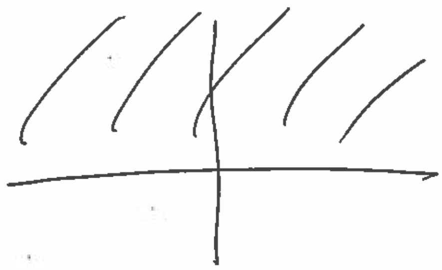
§2 Length & distance in ~~the~~ hyperbolic geometry

We will often swap between $\mathbb{R}^2$ and $\mathbb{C}$

$$(x,y) \in \mathbb{R}^2 \longleftrightarrow x+iy \in \mathbb{C}.$$

Defn The upper half-plane is

$$\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$$

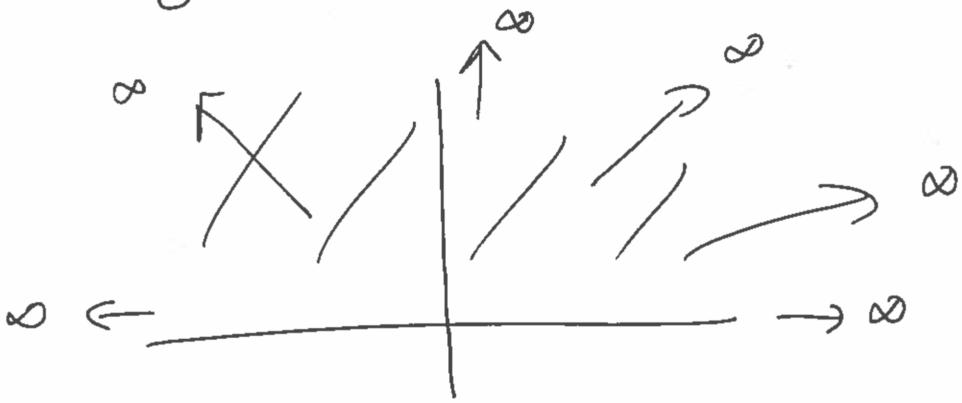


Defn The circle at infinity or boundary of $\mathbb{H}$ is

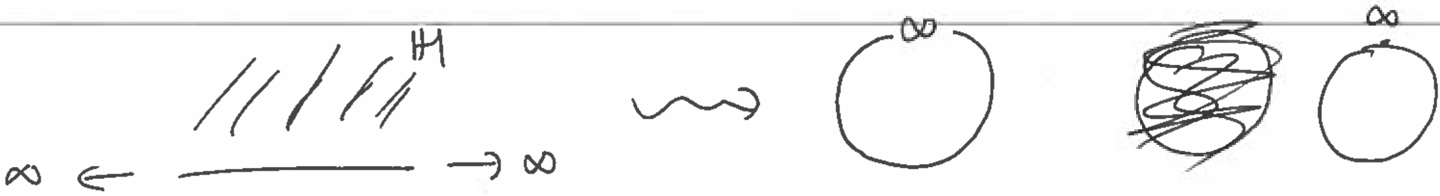
$$\partial \mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im} z = 0 \} \cup \{\infty\}.$$

What does ∞ mean? It's a point we've invented so that it makes sense to divide by 0:

$$\frac{1}{0} = \infty, \quad \lim_{x \to \infty} \frac{1}{x} = 0, \quad \lim_{x \to 0} \frac{1}{x} = \infty, \quad \frac{1}{\infty} = 0, \text{ etc}$$

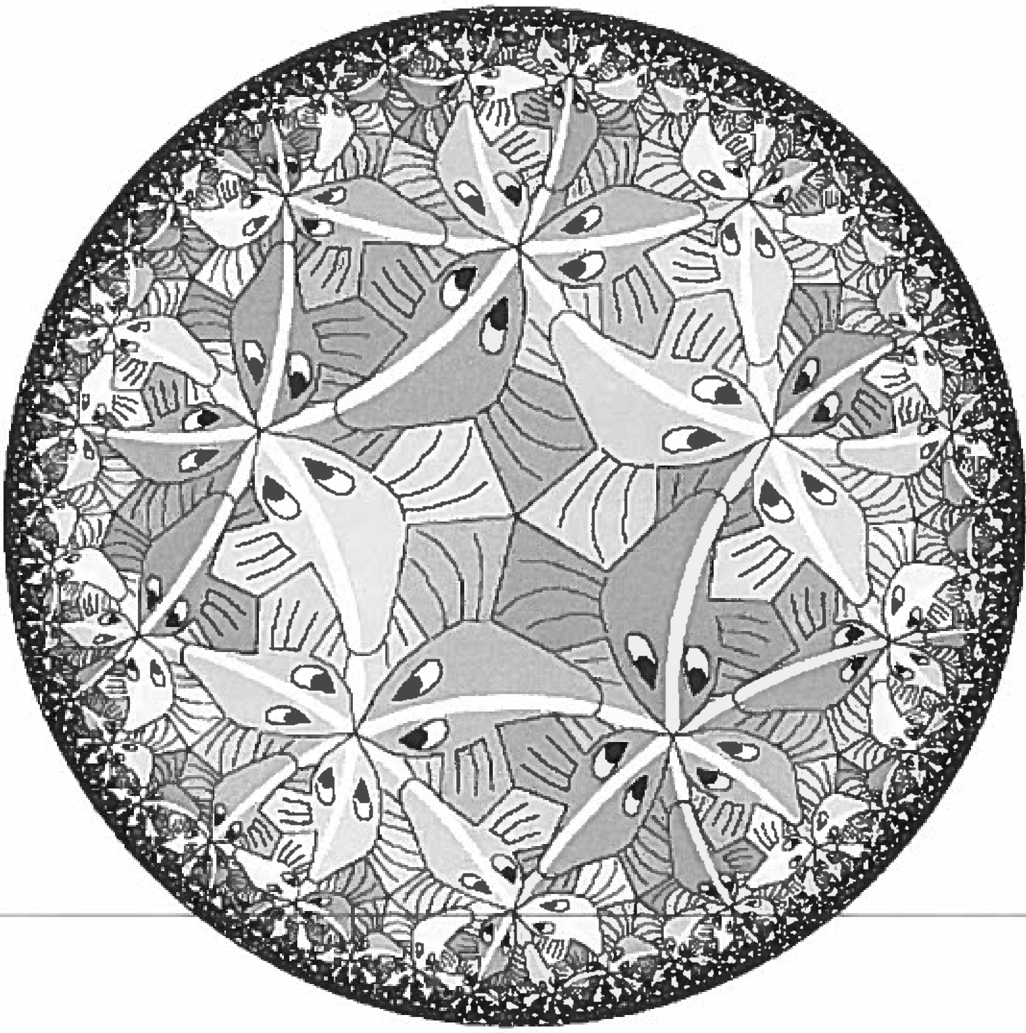


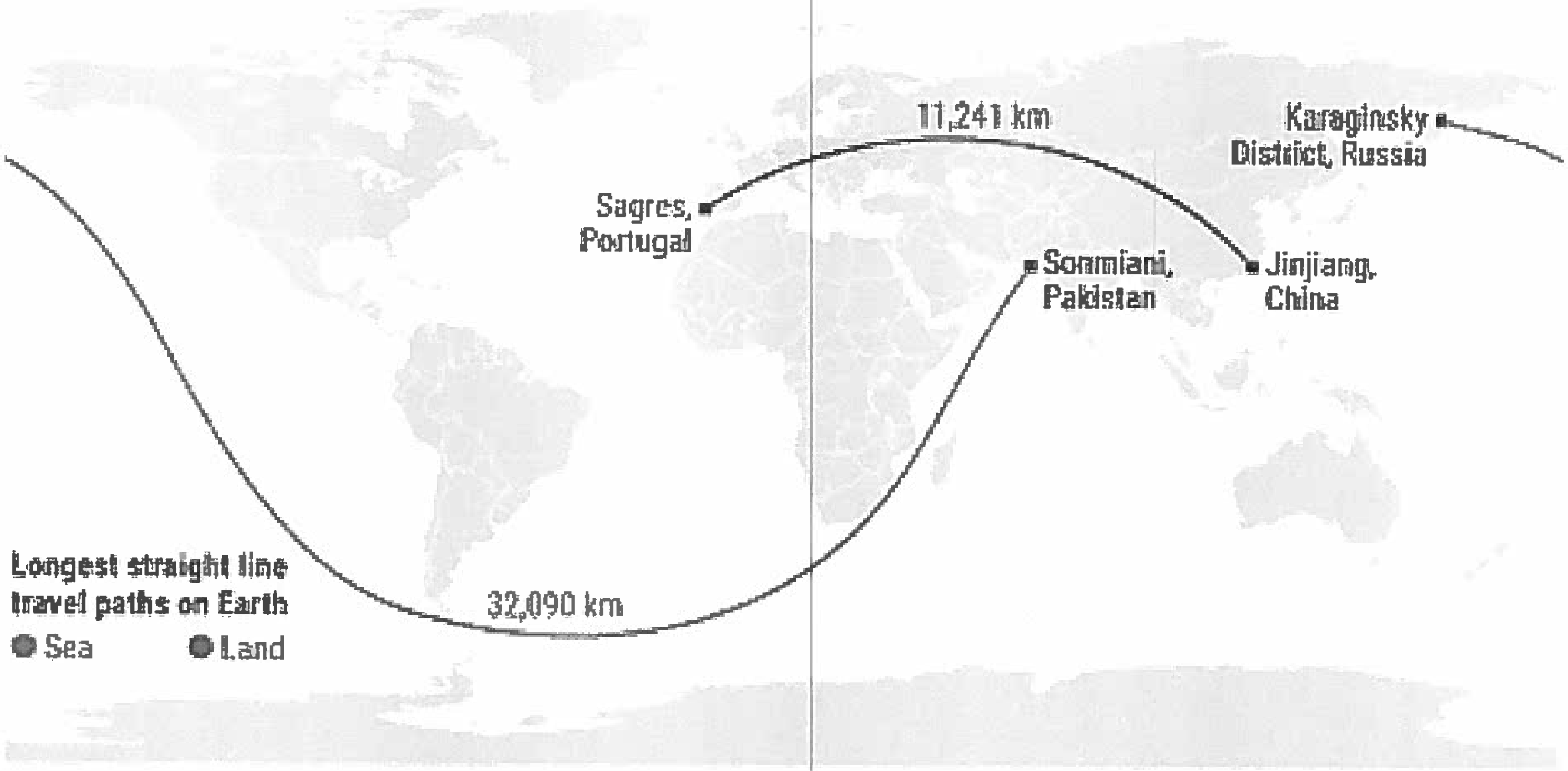
Why is $\partial \mathbb{H}$ a circle?

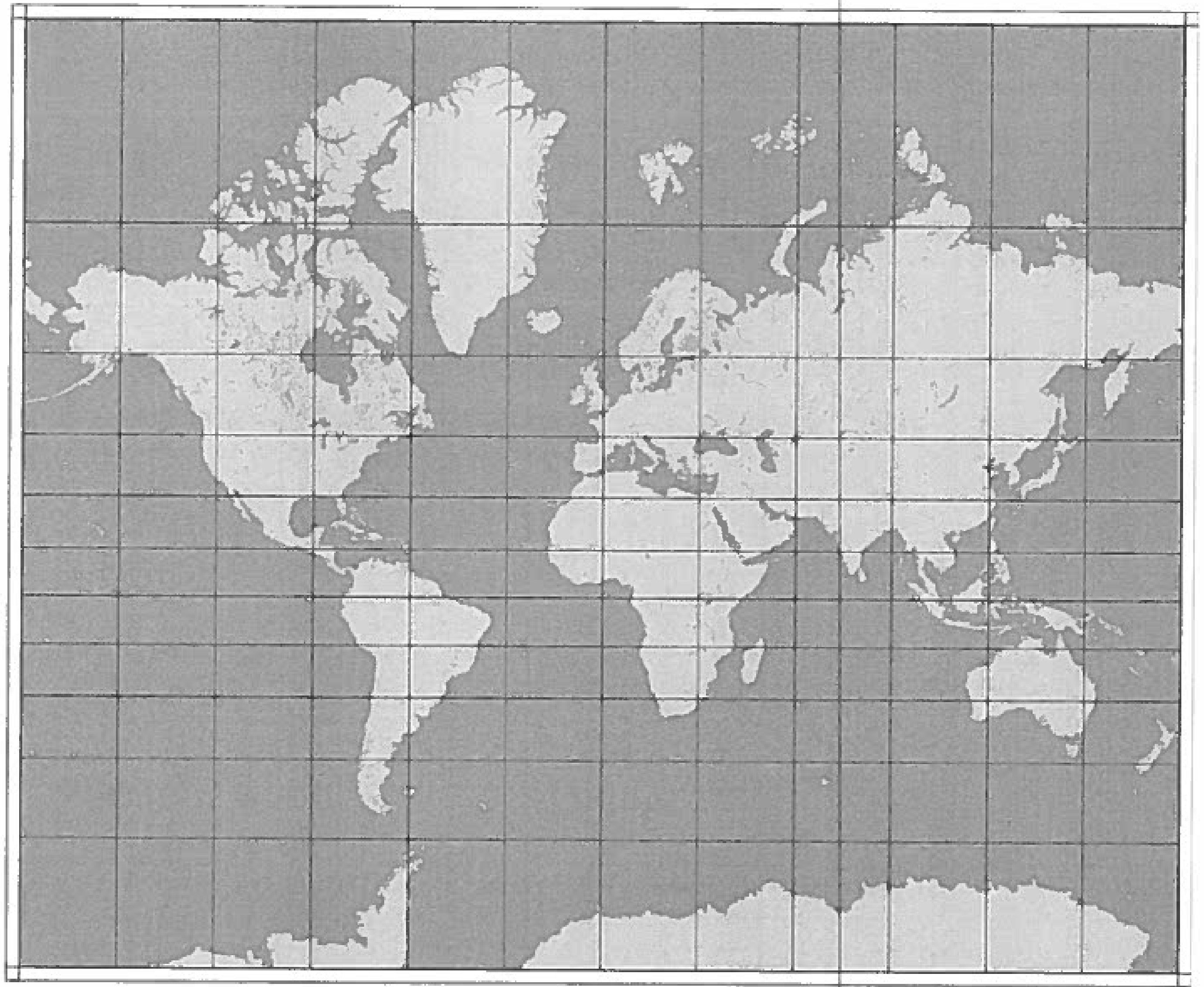


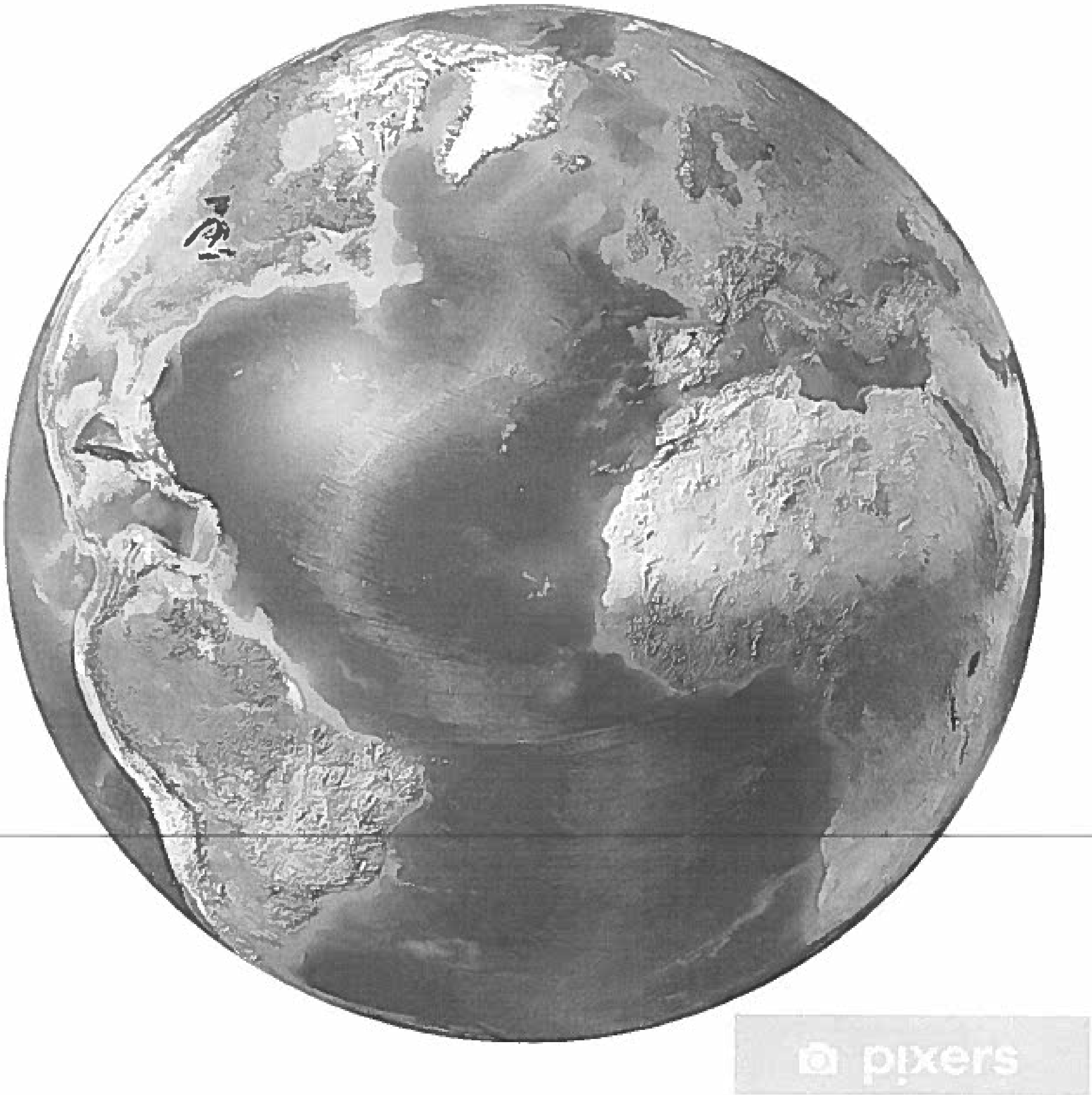
Why is $\partial \mathbb{H}$ the circle at infinity?

We will see that points in $\partial \mathbb{H}$ are an infinite ~~distance~~ hyperbolic distance away from any point in $\mathbb{H}$.




11,241 km
Karaginsky District, Russia
Sagres, Portugal
Sonmiani, Pakistan
Jinjiang, China
Longest straight line travel paths on Earth
Sea
Land
32,090 km

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What did we do yesterday?

- (not much...)
- Defined the upper half-plane

$$\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$$

and the boundary of $\mathbb{H}$:

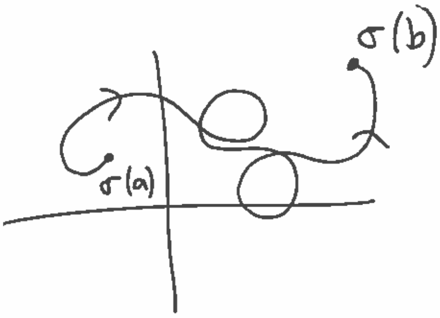
$$\partial\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) = 0\} \cup \{\infty\}.$$

What will we do today?

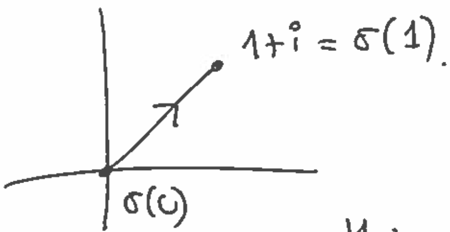
- Define the hyperbolic length of a path in $\mathbb{H}$
- Use this to define the hyperbolic distance between two points in $\mathbb{H}$
- Find an equation to describe circles/lines in $\mathbb{C}$

Path integrals. A *path* in $\mathbb{C}$ is the image of a continuous function $\sigma(\cdot) : [a,b] \to \mathbb{C}$. The function $\sigma(\cdot)$ is called a parametrisation of σ.

The end points of σ are $\sigma(a)$, $\sigma(b)$.

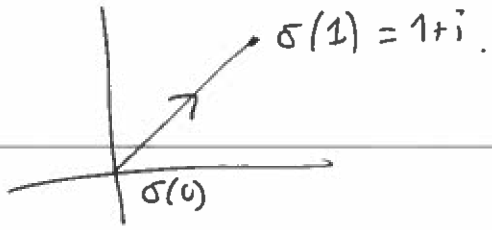


Example: $\sigma(t) = t + it \quad 0 \le t \le 1.$



Different parametrisations can give the same path:

$\sigma(t) = t^2 + it^2 \quad 0 \le t \le 1.$



If σ is differentiable and $f : \mathbb{C} \to \mathbb{R}$ is continuous then we define

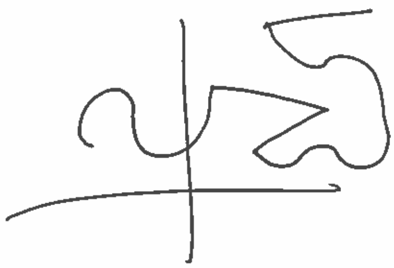
$$\int_\sigma f := \int_a^b f(\sigma(t))\, |\sigma'(t)|\, dt$$

$$\left(\text{here } |\sigma'(t)| = \sqrt{(\operatorname{Re} \sigma'(t))^2 + (\operatorname{Im} \sigma'(t))^2}. \right)$$

$$\sigma : [a,b] \to \mathbb{C}.$$

This is independent of the choice of parametrisation.

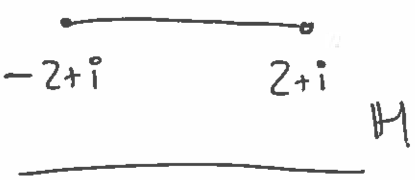
σ is piecewise differentiable (p/w diff'ble) if it comprises finitely many differentiable parts (equivalently, there are finitely many corners)



Defn Let $\sigma : [a,b] \to \mathbb{H}$ be a path in $\mathbb{H}$. Then

$$\text{length}_{\mathbb{H}}(\sigma) := \int_\sigma \frac{1}{\operatorname{Im} z} = \int_a^b \frac{1}{\operatorname{Im} \sigma(t)} |\sigma'(t)|\, dt$$

Example $\sigma(t) = -2 + 4t + i$
$0 \leq t \leq 1.$

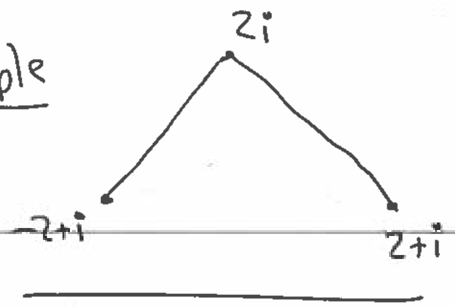


$\sigma'(t) = 4 \qquad |\sigma'(t)| = 4$

$\operatorname{Im} \sigma(t) = 1.$

$$\text{length}_{\mathbb{H}}(\sigma) = \int_0^1 \frac{1}{\operatorname{Im} \sigma(t)} |\sigma'(t)|\, dt = \int_0^1 4\, dt = 4$$

Example



$$\sigma(t) = \begin{cases} (2t-2) + i(1+t) & 0 \leq t \leq 1 \\ (2t-2) + i(3-t) & 1 \leq t \leq 2. \end{cases}$$

$$|\sigma'(t)| = \begin{cases} |2+i| & 0 \leq t \leq 1 \\ |2-i| & 1 \leq t \leq 2. \end{cases} = \sqrt{5}$$

$$\operatorname{Im} \sigma(t) = \begin{cases} 1+t & 0 \leq t \leq 1 \\ 3-t & 1 \leq t \leq 2. \end{cases}$$

$$\text{length}_{\mathbb{H}}(\sigma) = \int_0^1 \frac{\sqrt{5}}{1+t}\, dt + \int_1^2 \frac{\sqrt{5}}{3-t}\, dt$$

$$= \sqrt{5} \log(1+t)\Big|_0^1 - \sqrt{5} \log(3-t)\Big|_1^2$$

$$= 2\sqrt{5} \log 2 \simeq 3.1 < 4.$$

This suggest that paths of shortest length (geodesics) in hyp. geom are very different to those in Euclidean geometry.

Defn. The <u>hyperbolic metric</u> on $\mathbb{H}$ is: for $z, z' \in \mathbb{H}$ ③

$$d_{\mathbb{H}}(z, z') = \inf \left\{ \text{length}_{\mathbb{H}}(\sigma) \;\middle|\; \begin{array}{l} \sigma \text{ is a p/w diff'ble} \\ \text{path, from } z \text{ to } z' \\ \text{in } \mathbb{H}. \end{array} \right\}$$

One can show that this is a metric

§3. Circles & lines in $\mathbb{C}$, Möbius txs

The equation of a straight line in $\mathbb{R}^2$ is $ax + by + c = 0$, $a, b, c \in \mathbb{R}$.

What is this in $\mathbb{C}$? Let $z = x + iy$.

Then $x = \dfrac{z + \bar{z}}{2}$, $y = \dfrac{z - \bar{z}}{2i}$. Hence

$$a\left(\frac{z+\bar{z}}{2}\right) + b\left(\frac{z-\bar{z}}{2i}\right) + c = 0$$

$$\left(\frac{a - ib}{2}\right) z + \left(\frac{a+ib}{2}\right)\bar{z} + c = 0$$

Let $\alpha = 0 \in \mathbb{R}$, $\beta = \dfrac{a - ib}{2} \in \mathbb{C}$, $\gamma = c \in \mathbb{R}$. Then the straight line has equation

$$\boxed{\alpha z \bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0 \quad \Big| \quad \alpha, \gamma \in \mathbb{R},\ \beta \in \mathbb{C}}$$

In the <u>SPECIAL CASE</u> when $\beta \in \mathbb{R}$ (so $\beta = \bar{\beta}$) we have $b = 0$, so the straight line is $ax + c = 0$, ie a <u>vertical</u> straight line

The equation of a circle in $\mathbb{C}$ with centre z_0 and radius r is $|z - z_0| = r$.

ie $|z - z_0|^2 = (z - z_0)\overline{(z - z_0)} = r^2$

ie $(z - z_0)(\overline{z} - \overline{z_0}) = r^2$

ie $z\overline{z} - \overline{z_0}\, z - z_0 \overline{z} + |z_0|^2 - r^2 = 0$

Let $\alpha = 1 \in \mathbb{R}$, $\beta = -\overline{z_0} \in \mathbb{C}$, $\gamma = |z_0|^2 - r^2 \in \mathbb{R}$

$$\boxed{\alpha z\overline{z} + \beta z + \overline{\beta}\,\overline{z} + \gamma = 0 \quad \Big| \quad \alpha, \gamma \in \mathbb{R},\ \beta \in \mathbb{C}}$$

In the SPECIAL CASE when $\beta \in \mathbb{R}$ (so $\beta = \overline{\beta}$) we have $z_0 \in \mathbb{R}$, so the circle has a real centre.

Prop Let A be either a straight line in $\mathbb{C}$ or a circle in $\mathbb{C}$. Then A has equation

$$\alpha z\overline{z} + \beta z + \overline{\beta}\,\overline{z} + \gamma = 0 \qquad \alpha, \gamma \in \mathbb{R},\ \beta \in \mathbb{C}$$

Prop Let A be either a vertical straight line in $\mathbb{C}$ or a circle in $\mathbb{C}$ with a real centre. Then A has equation

$$\alpha z\overline{z} + \beta z + \beta\overline{z} + \gamma = 0 \qquad \alpha, \beta, \gamma \in \mathbb{R}.$$

Rmk When $\alpha = 0 \longrightarrow$ get a straight line
$\alpha \neq 0 \longrightarrow$ get a circle (divide through by α so that wlog $\alpha = 1$).