Solutions for MATH32052 Hyperbolic Geometry Coursework Solutions, May/June 2019

General feedback. There were a lot of high marks on the exam and (so as to be fair on students who hadn't taken this course) the exam board decided to scale the marks down slightly (73 was scaled to 70, but other marks were left alone). I was very pleased with how you performed on the course and I hope that you enjoyed it.

Question Q1(i)

Learning Outcome

Classify Möbius transformations in terms of their actions on the hyperbolic plane: (i), (ii) low level; (iii), (iv) medium level; (v) high level.
(i) is bookwork. The definitions in (ii) are bookwork, the examples are similar to exercises. (iii) is from the exercise sheets. (iv) was sketched in the lectures with details left as an exercise. (v) is unseen.

Solution

(i) A Möbius transformation is a transformation of the form $\gamma(z) = (az+b)/(cz+d)$, $a, b, c, d \in \mathbb{R}$, $ad - bc > 0$. [2 marks]

> **Feedback:** You need to remember to include the conditions that a, b, c, d are real and $ad - bc > 0$; most people did, but a small number did not.

(ii) Let $\gamma(z) = (az+b)/(cz+d)$. By dividing numerator and denominator by $\sqrt{ad-bc}$ there is no loss in generality in assuming that $ad - bc = 1$. When $ad - bc = 1$ we define $\tau(\gamma) = (a+d)^2$. [2 marks]

γ is hyperbolic precisely when $\tau(\gamma) > 4$.

γ is parabolic precisely when $\tau(\gamma) = 4$.

γ is elliptic precisely when $\tau(\gamma) \in [0, 4)$. [3 marks]

> **Feedback:** Common mistakes included saying that γ is elliptic if $\tau(\gamma) \in (0, 4)$. The case $\tau(\gamma) = 0$ can occur: eg $\gamma(z) = -1/z$. Several people proved the above classification; the question doesn't ask for it, so there's no point.

γ_1 is not normalised. In normalised form we have

$$\gamma_1(z) = \frac{\frac{7}{3}z - \frac{8}{3}}{\frac{2}{3}z - \frac{1}{3}}.$$

Hence $\tau(\gamma_1) = (7/3 - 1/3)^2 = 4$ so that γ_1 is parabolic.

γ_2 is not normalised. In normalised form we have

$$\gamma_2(z) = \frac{\frac{2}{3}z - \frac{1}{3}}{\frac{-7}{3}z + \frac{8}{3}}.$$

Hence $\tau(\gamma_1) = (2/3 + 8/3)^2 = \frac{100}{9} > 4$ so that γ_1 is hyperbolic.

> **Feedback:** All of these are straightforward calculations. Around 90% of people remembered to normalise Möbius transformations first, which I'm viewing as one of my greatest accomplishments this year :-)

[4 marks]

(iii) Let $\gamma_1 = (a_1z + b_1)/(c_1z + d_1)$ and $\gamma_2 = (a_2z + b_2)/(c_2z + d_2)$. Then their composition is

$$\begin{aligned} \gamma_2\gamma_1(z) &= \frac{a_2\left(\frac{a_1z+b_1}{c_1z+d_1}\right) + b_2}{c_2\left(\frac{a_1z+b_1}{c_1z+d_1}\right) + d_2} \\ &= \frac{(a_2a_1 + b_2c_1)z + (a_2b_1 + b_2d_1)}{(c_2a_1 + d_2c_1)z + (c_2b_1 + d_2d_1)}, \end{aligned}$$

which is a Möbius transformation of $\mathbb{H}$ as

$$\begin{aligned} &(a_2a_1 + b_2c_1)(c_2b_1 + d_2d_1) - (a_2b_1 + b_2d_1)(c_2a_1 + d_2c_1) \\ &= (a_1d_1 - b_1c_1)(a_2d_2 - b_2c_2) > 0. \end{aligned}$$

[Full credit will also be given if an explanation using the connection between composition of Möbius transformations and multiplication of matrices is given.] [6 marks]

Feedback: A straightforward calculation. A common omission was to forget to check that the 'new' $ad - bc$ satisfied $ad - bc > 0$. A few people mis-read the question and assumed that γ_1, γ_2 in this part of the question were the same as the explicit γ_1, γ_2 in part (ii); the question makes it clear that γ_1, γ_2 are arbitrary Möbius transformations. Some of you did this question by exploiting the connection between composition of Möbius transformations and multiplication of matrices; this was fine. provided that you stated that this is what you were doing.

(iv) Suppose that z satisfies $\alpha z\bar{z} + \beta z + \beta\bar{z} + \gamma = 0$. Let $w = \gamma(z)$ where $\gamma(z) = (az + b)/(cz + d)$. Then $z = (dw - b)/(-cw + a)$. Hence w satisfies

$$\alpha \frac{dw - b}{-cw + a} \frac{d\bar{w} - b}{-c\bar{w} + a} + \beta \frac{dw - b}{-cw + a} + \beta \frac{d\bar{w} - b}{-c\bar{w} + a} + \gamma = 0$$

i.e.

$$\alpha(dw-b)(d\bar{w}-b)+\beta(dw-b)(-c\bar{w}+a)+\beta(d\bar{w}-b)(-cw+a)+\gamma(-cw+a)(-c\bar{w}+a) = 0$$

Hence

$$\alpha' w\bar{w} + \beta' w + \beta' \bar{w} + \gamma' = 0$$

with

$$\begin{aligned} \alpha' &= \alpha d^2 - 2\beta dc + \gamma c^2 \\ \beta' &= -\alpha bd + \beta ad + \beta bc - \gamma ac \\ \gamma' &= \alpha b^2 - 2\beta ab + \gamma a^2 \end{aligned}$$

which has the same form as above. [8 marks]

Feedback: This is a proof that we did in the lectures (see Proposition 3.5.2 in the lecture notes).

(v) By the fact, γ maps circles to straight lines or to circles. Moreover γ maps $\partial\mathbb{H}$ to $\partial\mathbb{H}$. Note that A intersects $\partial\mathbb{H}$ at exactly one point. Hence $\gamma(A)$ must be either a straight line or a circle that intersects $\partial\mathbb{H}$ at exactly one point. In the first case, $\gamma(A)$ must be a horizontal straight line (which intersects $\partial\mathbb{H}$ at ∞ only; any other straight line will meet $\partial\mathbb{H}$ at two points). In the second case, $\gamma(A)$ must be a circle that touches the real axis at a single point. [5 marks]

Feedback: This is something that you haven't seen before. Of the (relatively few) answers given to this, a common mistake was to write that an arbitrary straight line meets $\partial\mathbb{H}$ in just one point (a point on the real axis), and not realise that it also meets $\partial\mathbb{H}$ at ∞.

Question Q2

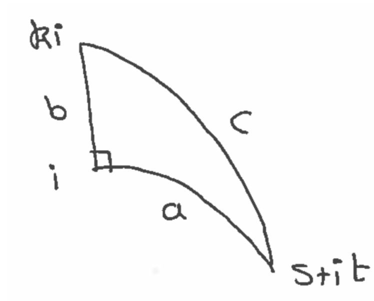
Learning Outcome

Prove results (Gauss-Bonnet Theorem, angle formulæ for triangles, etc as listed in the syllabus) in hyperbolic trigonometry and use them to calculate angles, side lengths, hyperbolic areas, etc, of hyperbolic triangles and polygons. (i), (ii) medium level. (iii) low level. (iv) high level.

(i) is bookwork. (ii) is from the exercise sheets. (iii) is bookwork. The first part of (iv) is a particular case of a result discussed in lectures; the second part is unseen.

Solution

(i) By applying a Möbius transformation of $\mathbb{H}$, we may assume that the vertex with internal angle $\pi/2$ is at i and that the side of length b lies along the imaginary axis; here we are using the fact that Möbius transformations are conformal. It follows that the side of length a lies along the geodesic given by the semi-circle centred at the origin with radius 1. Therefore, the other vertices of Δ can be taken to be at ki for some $k > 0$ and at $s + it$, where $s + it$ lies on the circle centred at the origin and of radius 1.



Using the formula for $\cosh d_{\mathbb{H}}(z,w)$ we have

$$\begin{aligned}
\cosh a &= 1 + \frac{|s+i(t-1)|^2}{2t} = 1 + \frac{s^2+(t-1)^2}{2t} = \frac{1}{t}, && (1)\\
\cosh b &= 1 + \frac{(k-1)^2}{2k} = \frac{1+k^2}{2k}, && (2)\\
\cosh c &= 1 + \frac{|s+i(t-k)|^2}{2tk} = 1 + \frac{s^2+(t-k)^2}{2tk} = \frac{1+k^2}{2tk}, && (3)
\end{aligned}$$

(using the fact that $s^2+t^2=1$). Hence $\cosh c = \cosh a \cosh b$. [8 marks]

Feedback: This is a proof from the course, see Theorem 5.7.1 in the lecture notes. The question asks you to explain why any reduction to the special case is possible (i.e. why applying a Möbius transformation doesn't change the internal angles or lengths); a common mistake was to not answer this. Some of you got lost in the algebra, often by not using the fact that $s^2+t^2=1$.

(ii) Construct a geodesic from vertex B to the geodesic segment $[A,C]$ in such a way that these geodesics meet at right-angles. This splits Δ into two right-angled triangles, BDA and BDC. Let the length of the geodesic segment $[B,D]$ be d, and suppose that BDA has internal angles $\beta_1, \pi/2, \alpha$ and side lengths d, b_1, c. Label BDC similarly.

We know that

$$\sin\beta_1 = \frac{\sinh b_1}{\sinh c},\ \cos\beta_1 = \frac{\tanh d}{\tanh c},\ \sin\beta_2 = \frac{\sinh b_2}{\sinh a},\ \cos\beta_2 = \frac{\tanh d}{\tanh a}.$$

By the hyperbolic version of Pythagoras' Theorem we know that

$$\cosh c = \cosh b_1 \cosh d,\ \cosh a = \cosh b_2 \cosh d.$$

Hence

$$\begin{aligned}
\sin\beta &= \sin(\beta_1+\beta_2)\\
&= \sin\beta_1\cos\beta_2 + \sin\beta_2\cos\beta_1\\
&= \frac{\sinh b_1}{\sinh c}\frac{\sinh d}{\cosh d}\frac{\cosh a}{\sinh a} + \frac{\sinh b_2}{\sinh a}\frac{\sinh d}{\cosh d}\frac{\cosh c}{\sinh c}\\
&= \frac{\sinh b_1 \sinh d}{\sinh c \sinh a}\cosh b_2 + \frac{\sinh b_2 \sinh d}{\sinh a \sinh c}\cosh b_1\\
&= \frac{\sinh d}{\sinh a \sinh c}(\sinh b_1 \cosh b_2 + \sinh b_2 \cosh b_1)\\
&= \frac{\sinh d}{\sinh a \sinh c}\sinh(b_1+b_2)\\
&= \frac{\sinh b \sinh d}{\sinh a \sinh c}.
\end{aligned}$$

Hence $\sin\alpha = \sinh d/\sinh c$ and $\sin\gamma = \sinh d/\sin a$. Substituting these into the above equality proves the result. [12 marks]

Feedback: This is Exercise 8.4.1 in the lecture notes.

(iii) Let Δ be a hyperbolic triangle with internal angles α, β and γ. Then $\mathrm{Area}_{\mathbb{H}} = \pi - (\alpha + \beta + \gamma)$. [2 marks]

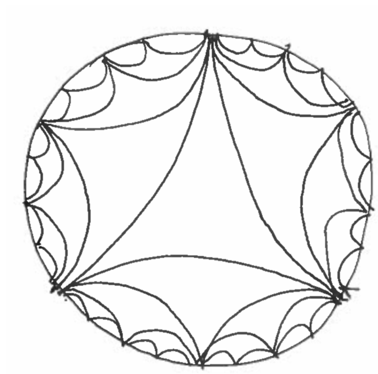
Feedback: I think almost everybody got this right: it's a standard (and very important!) result in the course.

(iv) Suppose the hyperbolic triangle Δ has internal angle α. As k polygons meet at each vertex, we must have that $k\alpha = 2\pi$, i.e. $\alpha = 2\pi/k$. By the Gauss-Bonnet Theorem

$$0 \leq \mathrm{Area}_{\mathbb{H}}(P) = \pi - 3 \times \frac{2\pi}{k}.$$

Hence $1 - 6/k > 0$, i.e. $k > 6$. Hence $k \geq 7$. [4 marks]

Feedback: Some of you wrote down (without proof) the condition that a tiling exists if and only if $1/n + 1/k < 1/2$. The question doesn't say that you can assume any result from the course, so if you use this fact then you would need to prove it (which is essentially going through the above calculation for an arbitrary n).



[4 marks]

Feedback: This was not well answered in general. The trick is to first think: what does an ideal triangle in the Poincaré disk look like? Once you've done that, you can then think what it would look like to have infinitely many of them meeting at each vertex.

Question Q3

Learning Outcome	**Solution**

Calculate a fundamental domain and a set of side-pairing transformations for a given Fuchsian group. (a)(i)–(iii) at low level. (b)(i), (ii), (iv) at medium level.

Compare different models (the upper half-plane model and the Poincaré disc model) of hyperbolic geometry. (b)(iii) at medium level.

(a)(i), (a)(ii) are bookwork. (a)(iii) is from the exercise sheets. (b)(i) is unseen. (b)(ii), (b)(iii), (b)(iv) are similar to exercise sheets.

(a) (i) $F \subset \mathbb{H}$ is a fundamental domain for Γ if (i) $\bigcup_{\gamma \in \Gamma} \gamma(\overline{F}) = \mathbb{H}$, (ii) $\gamma_1(F) \cap \gamma_2(F) = \emptyset$ if $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$.

[2 marks]

Feedback: This is a standard definition from the course. Remember that in (i) you need to take the union (not intersection!) over *all* $\gamma \in \Gamma$, and not over $\gamma \in \Gamma \setminus \{\mathrm{id}\}$.

(ii) Choose a point $p \in \mathbb{H}$ such that $\gamma(p) \neq p$ for all $\gamma \in \Gamma \setminus \{\mathrm{id}\}$.
For each $\gamma \in \Gamma \setminus \{\mathrm{id}\}$, construct the arc of geodesic $[p, \gamma(p)]$ from p to $\gamma(p)$.
Let $L_p(\gamma)$ denote the perpendicular bisector of $[p, \gamma(p)]$. Then $L_p(\gamma)$ divides $\mathbb{H}$ into two half-planes. One of these half-planes contains p; call this half-plane $H_p(\gamma)$.
Then $D(p) := \bigcap_{\gamma \in \Gamma \setminus \{\mathrm{id}\}} H_p(\gamma)$ is a Dirichlet region for Γ. [6 marks]

Feedback: This is a standard piece of bookwork from the course. Common mistakes were mostly in the final step: it's the intersection (not union!) over all *non-trivial* group elements in Γ.

(iii) Write $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$. Then $z = x + iy$ is on the perpendicular bisector of $[z_1, z_2]$ if and only if $d_{\mathbb{H}}(z, z_1) = d_{\mathbb{H}}(z, z_2)$ if and only if $\cosh d_{\mathbb{H}}(z, z_1) = \cosh d_{\mathbb{H}}(z, z_2)$ if and only if

$$1 + \frac{|z - z_1|^2}{2 \operatorname{Im} z \operatorname{Im} z_1} = 1 + \frac{|z - z_2|^2}{2 \operatorname{Im} z \operatorname{Im} z_2},$$

i.e.

$$\frac{|z - z_1|^2}{2yy_1} = \frac{|z - z_2|^2}{2yy_2}.$$

Simplifying this gives the claimed expression. [4 marks]

Feedback: This is Exercise 14.2(i).

(b) (i) Let $z = x + iy$ be on the perpendicular bisector of $[i, i + b]$. Then

$$|(x + iy) - i|^2 = |(x + iy) - (i + b)|^2$$

i.e.

$$\begin{aligned} x^2 + (y-1)^2 &= (x-b)^2 + (y-1)^2 \\ x^2 &= x^2 - 2xb + b^2 \\ 2xb &= b^2 \\ x &= b/2, \end{aligned}$$

so the perpendicular bisector is the vertical straight line with real part $b/2$. [4 marks]

Feedback: This is a straightforward calculation using (a)(iii) above, proving rigorously something that we said was 'clear from the geometry' (i.e. stare at the picture and it looks right) in the course.

(ii) Let $\gamma_n(z) = z + 4n$. Let $p = i$. By (ii) we have that $L_p(\gamma_n)$ is the vertical straight line with real part $2n$. Hence

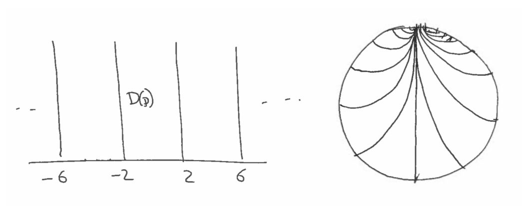
$$H_p(\gamma_n) = \begin{cases} \{z \in \mathbb{H} \mid \mathrm{Re}(z) < 2n\} & \text{if } n \geq 0 \\ \{z \in \mathbb{H} \mid \mathrm{Re}(z) > 2n\} & \text{if } n \leq 0. \end{cases}$$

Hence

$$D(p) = \bigcap_{n \neq 0} H_p(\gamma_n) = \{z \in \mathbb{H} \mid -2 < \mathrm{Re}(z) < 2\}.$$

[6 marks]

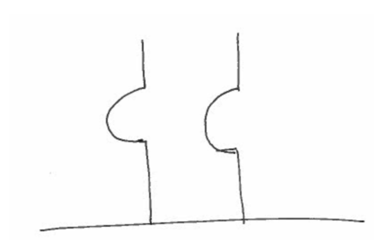
Feedback: We did the (very similar) case of integer translations in the lectures.

(iii) 

[4 marks]

Feedback: A very common mistake was to draw the strips to have horizontal width 2 (not 4).

(iv) $D(p)$ for an arbitrary choice of $p \in \mathbb{H}$ will be a vertical strip of width 4.
An example of a fundamental domain for Γ that is not of the form $D(p)$ for some $p \in \mathbb{H}$ is illustrated below.



[Any similar repeating shape that isn't a vertical strip is acceptable.] [4 marks]

Feedback: We did a very similar example to this in the lectures in the case of integer translations.

Learning Outcome

Use Poincaré's Theorem to construct examples of Fuchsian groups and calculate presentations in terms of generators and relations. (i) at low level. (ii) at medium level. (iii) at high level.

Relate the signature of a Fuchsian group to the algebraic and geometric properties of the Fuchsian group and to the geometry of the corresponding hyperbolic surface.. (iv) definition at low level, remainder at high level.

(i) is bookwork; (ii), (iii) are similar to exercise sheets. The definitions in (iv) are bookwork, the remainder is unseen.

Solution

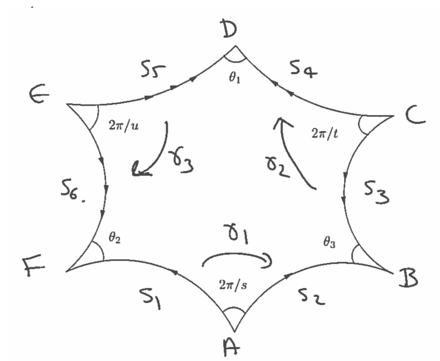
(i) The angle sum is $\mathrm{sum}(\mathcal{E}) := \sum_{j=1}^{n} \angle v_j$.

$\mathcal{E}$ satisfies the elliptic cycle condition if there exists an integer $n \geq 1$ such that $\mathrm{sum}(\mathcal{E}) = 2\pi/n$.

$\mathcal{E}$ is an accidental cycle if $\mathrm{sum}(\mathcal{E}) = 2\pi$. [4 marks]

Feedback: These are standard definitions from the course.

(ii) Label the diagram as below.



The elliptic cycles are:

(1)

$$\begin{pmatrix} A \\ s_1 \end{pmatrix} \xrightarrow{\gamma_1} \begin{pmatrix} A \\ s_2 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} A \\ s_1 \end{pmatrix}$$

elliptic cycle: $\mathcal{E}_1 = A$.
elliptic cycle transformation: γ_1.
angle sum: $\mathrm{sum}(\mathcal{E}_1) = 2\pi/n_1$.

(2)

$$\begin{pmatrix} C \\ s_3 \end{pmatrix} \xrightarrow{\gamma_2} \begin{pmatrix} C \\ s_4 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} C \\ s_3 \end{pmatrix}$$

elliptic cycle: $\mathcal{E}_2 = C$.
elliptic cycle transformation: γ_2.
angle sum: $\mathrm{sum}(\mathcal{E}_2) = 2\pi/n_2$.

(3)

$$\begin{pmatrix} E \\ s_5 \end{pmatrix} \xrightarrow{\gamma_3} \begin{pmatrix} E \\ s_6 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} E \\ s_5 \end{pmatrix}$$

elliptic cycle: $\mathcal{E}_3 = E$.
elliptic cycle transformation: γ_3.
angle sum: $\mathrm{sum}(\mathcal{E}_3) = 2\pi/n_3$.

(4)

$$\begin{array}{cccc} \begin{pmatrix} B \\ s_3 \end{pmatrix} & \xrightarrow{\gamma_2} & \begin{pmatrix} D \\ s_4 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} D \\ s_5 \end{pmatrix} \\ & \xrightarrow{\gamma_3} & \begin{pmatrix} F \\ s_6 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} F \\ s_1 \end{pmatrix} \\ & \xrightarrow{\gamma_1} & \begin{pmatrix} B \\ s_2 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} B \\ s_3 \end{pmatrix} \end{array}$$

elliptic cycle: $\mathcal{E}_4 = B \to D \to F$.
elliptic cycle transformation: $\gamma_1\gamma_3\gamma_2$.
angle sum: $\mathrm{sum}(\mathcal{E}_4) = \theta_1 + \theta_2 + \theta_3$.

[12 marks]

> **Feedback:** Once you've got the hang of (vertex,side)-chasing, this is very straightforward.

(iii) The elliptic cycle condition holds for $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ when s, t, u are integers, say $s = n_1, t = n_2, u = n_3$. Note that the question implies that $n_1, n_2, n_3 \geq 2$.

If $\theta_1 + \theta_2 + \theta_3 = 2\pi/n$ for some $n \geq 1$ then the elliptic cycle condition holds for $\mathcal{E}_4$ with order n.

Hence $\gamma_1, \gamma_2, \gamma_3$ generate a Fuchsian group Γ and

$$\Gamma = \langle a, b, c \mid a^{n_1} = b^{n_2} = c^{n_3} = (acb)^n = e \rangle.$$

[6 marks]

> **Feedback:** The point of this is that one needs s, t, u to be integers (the question just says that they are real numbers ≥ 2) for the ECC to hold; several people ignored this point. We also need n (as defined above) to be an integer.

(iv) Let Γ be a cocompact Fuchsian group. Let $\mathcal{E}_1, \ldots, \mathcal{E}_r$ be the non-accidental elliptic cycles and let $\mathcal{E}_{r+1}, \ldots, \mathcal{E}_s$ be the accidental cycles. Let g be the genus of $\mathbb{H}/\Gamma$. Then $\mathrm{sig}(\Gamma) = (g; m_1, \ldots, m_r)$. (We write $(-; m_1, \ldots, m_r)$ if $g = 0$.)

> **Feedback:** The signature is a definition from the course; quite a few people didn't write down the definition and went straight into calculating the genus, etc.

In the example above, the hexagon forms a triangulation of $\mathbb{H}/\Gamma$ with $V = 4$ vertices (=number of elliptic cycles), $E = 3$ sides, and $F = 1$ faces. Hence by Euler's formula

$$2 - 2g = \chi(\mathbb{H}/\Gamma) = V - E + F = 4 - 3 + 1 = 2.$$

Hence $g = 0$.

> **Feedback:** Alternatively, think about what happens when you the paired sides together. You get something that looks like a drawstring bag, and so has genus $g = 0$.

Hence

$$\mathrm{sig}(\Gamma) = (-; n_1, n_2, n_3, n).$$

$\mathbb{H}/\Gamma$ is a topological sphere with 4 marked points. [8 marks]