2 hours

THE UNIVERSITY OF MANCHESTER

HYPERBOLIC GEOMETRY

?? May/June 2019

??:?? – ??:??

Answer **THREE** of the **FOUR** questions.
If all four questions are attempted then credit will be given for the three best answers.

Electronic calculators may be used, provided that they cannot store text.

Notation: Throughout, $\mathbb{H}$ denotes the upper half-plane, $\partial\mathbb{H}$ denotes the boundary of $\mathbb{H}$, $\mathbb{D}$ denotes the Poincaré disc, and $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$.

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1.

(i) What does it mean to say that γ is a Möbius transformation of $\mathbb{H}$?

[2 marks]

(ii) Let $\gamma \in \text{Möb}(\mathbb{H})$. How is $\tau(\gamma)$ defined? Briefly state (without proof) how to use $\tau(\gamma)$ to determine if γ is hyperbolic, parabolic or elliptic.

[5 marks]

Let

$$\gamma_1(z) = \frac{7z-8}{2z-1}, \quad \gamma_2(z) = \frac{2z-1}{-7z+8}.$$

Determine whether each of γ_1 and γ_2 is hyperbolic, parabolic or elliptic.

[4 marks]

(iii) Suppose that $\gamma_1, \gamma_2 \in \text{Möb}(\mathbb{H})$. Prove that the composition $\gamma_1\gamma_2 \in \text{Möb}(\mathbb{H})$.

[6 marks]

(iv) It was proved in the course that, in $\mathbb{C}$, vertical straight lines and circles with real centres have equations of the form

$$\alpha z\bar{z} + \beta z + \beta\bar{z} + \gamma = 0$$

where $\alpha, \beta, \gamma \in \mathbb{R}$.

Suppose that A is either a vertical straight line or a circle with a real centre and that $\gamma \in \text{Möb}(\mathbb{H})$. Show that $\gamma(A)$ is also either a straight line or a circle in $\mathbb{C}$.

[8 marks]

(v) One can also prove the following fact (you do not need to prove this yourself): Let A be a straight line (not necessarily vertical) or a circle (not necessarily with a real centre). Let $\gamma \in \text{Möb}(\mathbb{H})$. Then $\gamma(A)$ is also either a straight line (not necessarily vertical) or a circle (not necessarily with a real centre).

Let A be a circle as illustrated in Figure 1.

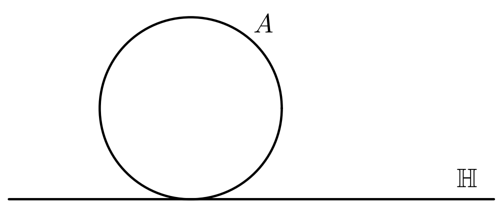


Figure 1: See Question 1(v).

Let $\gamma \in \text{Möb}(\mathbb{H})$ be a Möbius transformation of $\mathbb{H}$. Using the above fact, describe (in words) the possibilities for $\gamma(A)$, justifying your answer.

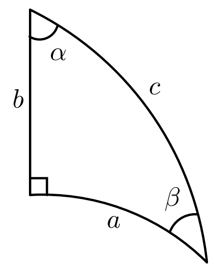
[5 marks]

2. Throughout this question you may use, without proof, the formula

$$\cosh d_{\mathbb{H}}(z, w) = 1 + \frac{|z-w|^2}{2\,\mathrm{Im}(z)\,\mathrm{Im}(w)}.$$

You may also assume that in a hyperbolic right-angled triangle with angles $\alpha, \beta, \pi/2$ and sides of hyperbolic length a, b, c (as illustrated below) we have

$$\sin\alpha = \frac{\sinh a}{\sinh c},$$
$$\cos\alpha = \frac{\tanh b}{\tanh c}.$$



You may also use the trig and hyperbolic trig identities: $\cosh^2 x - \sinh^2 x = 1$, $\sinh 2x = 2\sinh x\cosh x$, $\sin(x+y) = \sin x\cos y + \cos x \sin y$, $\sinh(x+y) = \sinh x\cosh y + \cosh x \sinh y$.

(i) Let Δ be an arbitrary hyperbolic right-angled triangle with sides of hyperbolic lengths a, b, c. Suppose that the side of hyperbolic length c is opposite the right-angle. Prove the Hyperbolic Pythagoras' Theorem:

$$\cosh c = \cosh a \cosh b.$$

(If you reduce Δ to a special case by applying a Möbius transformation then you should briefly explain why you can do this.)

[8 marks]

(ii) Let Δ be an arbitrary acute hyperbolic triangle with internal angles α, β, γ and opposite sides of hyperbolic length a, b, c (see Figure 2). Prove the hyperbolic sine rule:

$$\frac{\sin\alpha}{\sinh a} = \frac{\sin\beta}{\sinh b} = \frac{\sin\gamma}{\sinh c}.$$

Hint: drop a perpendicular from a vertex to the opposite side to create two right-angled triangles.

[12 marks]

(iii) Let Δ be a hyperbolic triangle with internal angles α, β and γ. State, without proof, the Gauss-Bonnet theorem for a hyperbolic triangle.

[2 marks]

(iv) Consider a tiling of the hyperbolic plane by regular hyperbolic triangles with k triangles meeting at each vertex. Use the Gauss-Bonnet Theorem to show that $k \geq 7$.

[4 marks]

(v) Draw a tiling of the Poincaré disc by regular hyperbolic triangles when $k = \infty$.

[4 marks]

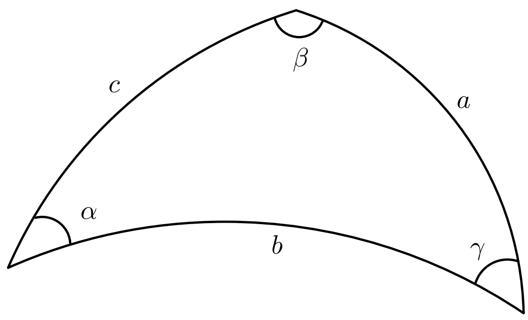


Figure 2: An acute hyperbolic triangle; see Question 2(ii).

3.

(a) (i) Let Γ be a Fuchsian group acting on $\mathbb{H}$. What does it mean to say that an open set $F \subset \mathbb{H}$ is a fundamental domain for Γ?

[2 marks]

(ii) Briefly explain how to construct a Dirichlet region for a Fuchsian group Γ. (Your answer should include a definition of the terms $[p, \gamma(p)]$, $L_p(\gamma)$, $H_p(\gamma)$, $D(p)$ that were defined in the lectures.)

[6 marks]

(iii) It was proved in the lectures that the perpendicular bisector of the arc of geodesic $[z_1, z_2]$ between two points $z_1, z_2 \in \mathbb{H}$ is given by

$$\{z \in \mathbb{H} \mid d_{\mathbb{H}}(z, z_1) = d_{\mathbb{H}}(z, z_2)\}. \tag{1}$$

Write $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Show that (1) can be written in the form

$$\{z \in \mathbb{H} \mid y_2|z - z_1|^2 = y_1|z - z_2|^2\}. \tag{2}$$

(You may use the fact that

$$\cosh d_{\mathbb{H}}(z, w) = 1 + \frac{|z - w|^2}{2\,\mathrm{Im}(z)\,\mathrm{Im}(w)}.)$$

[4 marks]

(b) (i) Let $\gamma(z) = z + b$ where $b \in \mathbb{R}$. Let $p = i$. Use the formula (2) to calculate the perpendicular bisector of $[p, \gamma(p)]$.

[4 marks]

(ii) Let $\Gamma = \{\gamma_n \mid \gamma_n(z) = z + 4n,\ n \in \mathbb{Z}\}$. Use the procedure you outlined in (a)(ii) and the result of (b)(i) above to show that $D(i) = \{z \in \mathbb{H} \mid -2 < \mathrm{Re}(z) < 2\}$.

[6 marks]

(iii) Draw the resulting tessellation in $\mathbb{H}$. What does this tessellation look like in $\mathbb{D}$?

[4 marks]

(iv) Write down what you think $D(p)$ will look like for an arbitrary choice of $p \in \mathbb{H}$.
Find a fundamental domain for Γ which is *not* of the form $D(p)$ for some $p \in \mathbb{H}$.

[4 marks]

4.

(i) Let $\mathcal{E} = v_1 \to v_2 \to \cdots \to v_n$ be an elliptic cycle. Let the internal angle at v_j be $\angle v_j$.

Define $\mathrm{sum}(\mathcal{E})$.

What does it mean to say that $\mathcal{E}$ satisfies the *elliptic cycle condition*?

What does it mean to say that $\mathcal{E}$ is an *accidental cycle*?

[4 marks]

(ii) Consider the hyperbolic hexagon illustrated in Figure 3 with the side-pairings indicated. Here $s, t, u \geq 2$ are real numbers and $\theta_1 + \theta_2 + \theta_3 \leq \pi$. (You may assume that such a hyperbolic hexagon exists.)

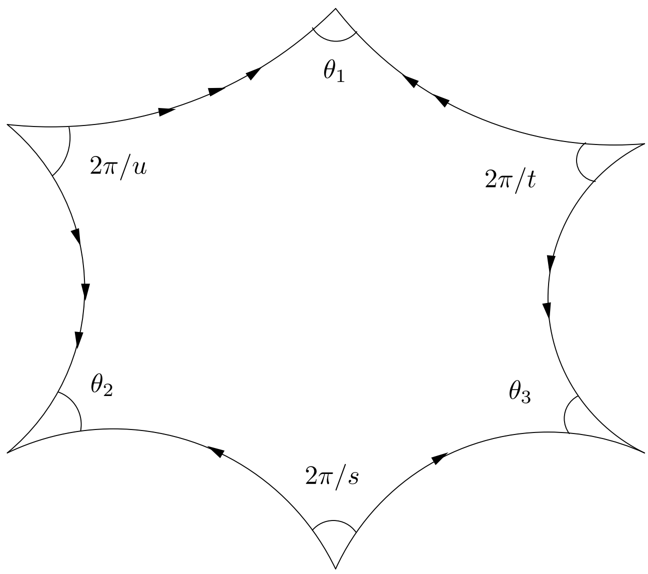


Figure 3: A hyperbolic hexagon; see Question 4(ii).

Show that there are 4 elliptic cycles. In each case, calculate the elliptic cycle transformations and the angle sum.

[12 marks]

(iii) Determine conditions on $\theta_1, \theta_2, \theta_3, s, t, u$ so that the side-pairing transformations generate a Fuchsian group Γ. Give a presentation for Γ in terms of generators and relations.

[6 marks]

(iv) How is the signature of a cocompact Fuchsian group defined?

Assume that $\theta_1, \theta_2, \theta_3, s, t, u$ are such that the side-pairing transformations in Figure 3 generate a Fuchsian group Γ. Use Euler's formula to calculate the genus g of $\mathbb{H}/\Gamma$. Hence write down the possibilities for the signature of Γ. Briefly describe $\mathbb{H}/\Gamma$ in each case.

[8 marks]

END OF EXAMINATION PAPER