Solutions for MATH32051 Hyperbolic Geometry Solutions and Feedback, January 2020

The learning outcomes for the course unit are:

- **ILO1**: calculate the hyperbolic distance between and the geodesic through points in the hyperbolic plane,
- **ILO2**: compare different models (the upper half-plane model and the Poincaré disc model) of hyperbolic geometry,
- **ILO3**: prove results (Gauss-Bonnet Theorem, angle formul for triangles, etc as listed in the syllabus) in hyperbolic trigonometry and use them to calculate angles, side lengths, hyperbolic areas, etc, of hyperbolic triangles and polygons,
- **ILO4**: classify Möbius transformations in terms of their actions on the hyperbolic plane,
- **ILO5**: calculate a fundamental domain and a set of side-pairing transformations for a given Fuchsian group,
- **ILO6**: define a finitely presented group in terms of generators and relations,
- **ILO7**: use Poincaré's Theorem to construct examples of Fuchsian groups and calculate presentations in terms of generators and relations,
- **ILO8**: relate the signature of a Fuchsian group to the algebraic and geometric properties of the Fuchsian group and to the geometry of the corresponding hyperbolic surface.

Question Q1(i)

Learning Outcome

ILO1: assessed at low level. Bookwork from the lectures.

Solutions

Let $z = x + iy$ so that $x = (z + \bar{z})/2$, $y = (z - \bar{z})/2i$. Then

$$\begin{aligned} ax + by + c = 0 \quad &\Leftrightarrow \quad a\left(\frac{z+\bar{z}}{2}\right) + b\left(\frac{z-\bar{z}}{2i}\right) + c = 0 \\ &\Leftrightarrow \quad \left(\frac{a-ib}{2}\right)z + \left(\frac{a+ib}{2}\right)\bar{z} + c = 0 \end{aligned}$$

which has the form $\beta z + \bar{\beta}\bar{z} + \gamma = 0$, $\beta \in \mathbb{C}, \gamma \in \mathbb{R}$.

Vertical straight lines have equations of the form $ax + c = 0$, i.e. $b = 0$. From the above, we then have that β is real and the equation has the form $\beta z + \beta\bar{z} + \gamma = 0$.

[4 marks]

Feedback. This was very well answered in general. Some people did it 'backwards' compared to the above (i.e. start with $\beta z + \bar{\beta}\bar{z} + \gamma = 0$ and rearrange this to get an equation of the form $ax + by + c = 0$; this is absolutely fine).

Question Q1(ii)

Learning Outcome

ILO1: assessed at low level. Bookwork from the lectures.

Solutions

We have

$$|z - z_0|^2 = r^2 \Leftrightarrow (z - z_0)(\bar{z} - z_0) = r^2 \Leftrightarrow z\bar{z} - \bar{z}_0 z - z_0\bar{z} + z_0^2 - r^2 = 0$$

which is of the form (*) (noting that $\bar{z}_0 = z_0$ as z_0 is real). We have $\beta = -\bar{z}_0$ and $\gamma = |z_0|^2 - r^2$.

When z_0 is real we have the equation

$$z\bar{z} - z_0 z - z_0\bar{z} + |z_0^2| - r^2 = 0$$

[6 marks]

Feedback. Note that the question explicitly asks you to give a formula for β, γ in terms of z_0 and r. Omitting this was a common source of losing 2 (easy) marks.

Question Q1(iii)

Learning Outcome
ILO1: assessed at medium level.

Similar to exercise sheets

Solutions
We know that $-5+12i$ and $12+5i$ both satisfy an equation of the form (*). Hence we have

$$\begin{aligned}(-5+12i)(-5-12i)+\beta(-5+12i)+\beta(-5-12i)+\gamma &= 0\\ (12+5i)(12-5i)+\beta(12+5i)+\beta(12-5i)+\gamma &= 0,\end{aligned}$$

equivalently

$$\begin{aligned}169-10\beta+\gamma &= 0\\ 169+24\beta+\gamma &= 0.\end{aligned}$$

Subtracting one equation from the other gives $\beta=0$. Hence $\gamma=-169$. Hence $-5+12i$ and $5+12i$ lie on the semi-circle with equation $z\bar{z}-169=0$, i.e. $|z|=13$.

The semi-circle has centre 0 and radius 13.

The geodesic has endpoints at -13 and 13. The Möbius transformation

$$\gamma(z)=\frac{z-13}{z+13}$$

maps $-13, 13$ to $0, \infty$ respectively and so maps this semi-circle to the imaginary axis.

The point $\frac{39}{5}+\frac{52}{5}i$ lies on the geodesic with end-points $-13, 13$. Hence $\gamma(\frac{39}{5}+\frac{52}{5}i)$ lies on the imaginary axis, say $\gamma(\frac{39}{5}+\frac{52}{5}i)=ki$ for some $k>0$. Let $\gamma_1(z)=kz$. Then $\gamma_1\gamma$ maps the geodesic with end-points $-13, 13$ to the imaginary axis and maps z_1 to i.

[8 marks]

Feedback. Some of you noticed that the points form a (5,12,13) Pythagorean triple and from this wrote down the equation $|z|=13$; this is absolutely fine. A common mistake was to write $\gamma(z)=(z+13)/(z-13)$ for the transformation that maps this geodesic to the imaginary axis. Note that this is not a Möbius transformation of $\mathbb{H}$ as '$ad-bc$' is negative.

Question Q1(iv)

Learning Outcome
ILO1: assessed at medium level.

Bookwork (proof from the lectures)

Solutions
Let $\sigma : [a,b] \to \mathbb{H}$, $\sigma(t)=it$. Then σ is a path from ia to ib. We have $|\sigma'(t)|=|i|=1$ and $\operatorname{Im}\sigma(t)=t$. Hence

$$\text{length}_{\mathbb{H}}(\sigma)=\int_a^b \frac{dt}{t}=\log t|_a^b=\log b/a.$$

Now suppose that $\sigma : [0,1] \to \mathbb{H}$ is an arbitrary path from ia to ib. Write $\sigma(t)=x(t)+iy(t)$. Note that $y(0)=a, y(1)=b$. We have

$$\begin{aligned}\text{length}_{\mathbb{H}}(\sigma) &= \int_0^1 \frac{\sqrt{x'(t)^2+y'(t)^2}}{y(t)}\,dt\\ &\geq \int_0^1 \frac{y(t)}{y'(t)}\,dt\\ &= \log y(t)|_0^1=\log b/a\end{aligned}$$

with equality if and only if $x'(t)=0$ and $y'(t)$ does not change sign. Hence $x(t)=\text{constant}=0$ and $y(t)$ strictly increases from a to b. Hence we have equality if and only if $\sigma(t)$ is the arc of imaginary axis from ia to ib.

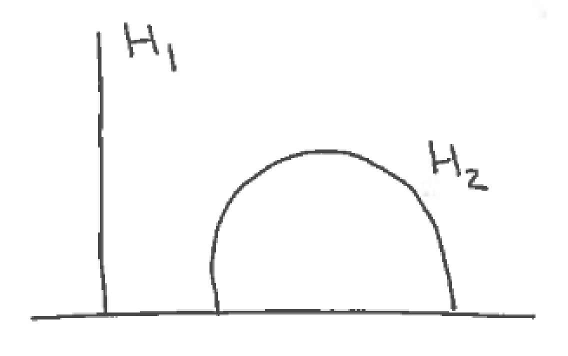
[8 marks]

Feedback. This is a proof from the course.

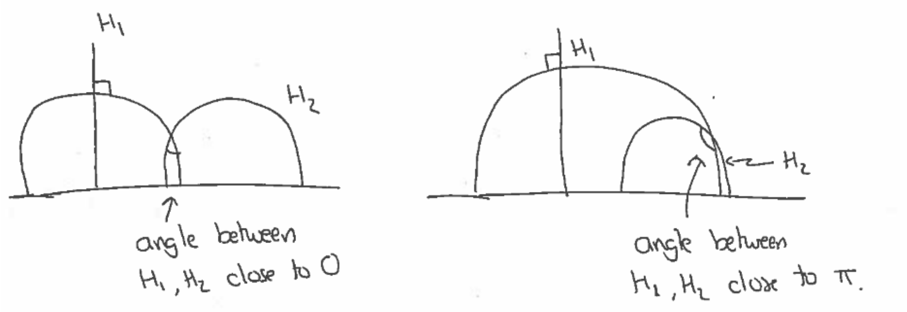
Question Q1(v)

Learning Outcome
ILO1: unseen

Solutions
Without loss of generality we can assume that H_1 is the imaginary axis and H_2 is a semi-circle with real centre that is disjoint from H_1, as illustrated.



Any geodesic that passes through H_1 at right-angles must do some horizontally. Consider geodesics that pass through H_1 horizontally and that intersect H_2; in particular consider the angle of intersection with H_2, as illustrated. The two extremes correspond to angles 0 and π, and this angle increases continuously as the geodesic changes. Hence by the Intermediate Value Theorem, there is a unique geodesic that intersects both H_1 and H_2 at right-angles.



[4 marks]

Feedback. There were few genuinely complete answers here (as to be expected: the last part of a question really should be stretching people's abilities!). I was pleased by the number of people who had the basic idea correct though.

Question Q2(i)

Learning Outcome
ILO4: assessed at a low level. Similar to example sheets.

Solutions

(a) This is a Möbius transformation of $\mathbb{D}$: take $\alpha = e^{i\theta/2}, \beta = 0$.
(b) This is not a Möbius transformation of $\mathbb{D}$: it has the form $\dfrac{0z+1}{-1z+0}$, so α would be 0 and $\beta = 1, \bar{\beta} = -1$.

[4 marks]

Feedback. For (a): a very common mistake was to write $\gamma(z) = \frac{e^{i\theta}z+0}{0z+1}$ so $\alpha = e^{i\theta}$ and $\beta = 0$. This isn't correct (the denominator doesn't have a $\bar{\alpha}$ term in it).
For (b): another way of seeing that this is not a Möbius trans-

formation of $\mathbb{D}$ is to note that $0 \in \mathbb{D}$ but $\gamma(0) = \infty \notin \mathbb{D}$.

Question Q2(ii)

Learning Outcome

ILO4: assessed at a medium level, similar to example sheets

Solutions

Let
$$\gamma_1(z) = \frac{\alpha_1 z + \beta_1}{\bar{\beta}_1 z + \bar{\alpha_1}}, \quad \gamma_2(z) = \frac{\alpha_2 z + \beta_2}{\bar{\beta}_2 z + \bar{\alpha_2}}.$$
Then
$$\begin{aligned}
\gamma_1\gamma_2(z) &= \gamma_1\left(\frac{\alpha_2 z + \beta_2}{\bar{\beta}_2 z + \bar{\alpha_2}}\right) \\
&= \frac{\alpha_1\left(\frac{\alpha_2 z + \beta_2}{\bar{\beta}_2 z + \bar{\alpha_2}}\right) + \beta_1}{\bar{\beta}_1\left(\frac{\alpha_2 z + \beta_2}{\bar{\beta}_2 z + \bar{\alpha_2}}\right) + \bar{\alpha_1}} \\
&= \frac{\alpha_1(\alpha_2 z + \beta_2) + \beta_1(\bar{\beta}_2 z + \bar{\alpha_2})}{\bar{\beta}_1(\alpha_2 z + \beta_2) + \bar{\alpha_1}(\bar{\beta}_2 z + \bar{\alpha_2})} \\
&= \frac{(\alpha_1\alpha_2 + \beta_1\bar{\beta}_2)z + (\alpha_1\beta_2 + \beta_1\bar{\alpha_2})}{(\bar{\beta}_1\alpha_2 + \bar{\alpha_1}\bar{\beta}_2)z + (\bar{\beta}_1\beta_2 + \bar{\alpha_1}\bar{\alpha_2})}.
\end{aligned}$$
This is a Möbius transformation of $\mathbb{D}$ as
$$\begin{aligned}
&(\alpha_1\alpha_2 + \beta_1\bar{\beta}_2)(\bar{\beta}_1\beta_2 + \bar{\alpha_1}\bar{\alpha_2}) - (\alpha_1\beta_2 + \beta_1\bar{\alpha_2})(\bar{\beta}_1\alpha_2 + \bar{\alpha_1}\bar{\beta}_2) \\
&= (\alpha_1\bar{\alpha_1} - \beta_1\bar{\beta}_1)(\alpha_2\bar{\alpha_2} - \beta_2\bar{\beta}_2) \\
&= (|\alpha_1|^2 - |\beta_1|^2)(|\alpha_2|^2 - |\beta_2|^2) > 0.
\end{aligned}$$

[8 marks]

Feedback. This is very similar to a calculation we did in the lectures/one of the tutorials.

Question Q2(iii)

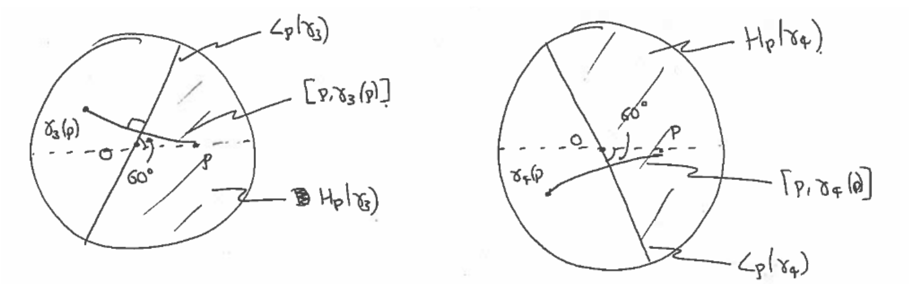
Learning Outcome

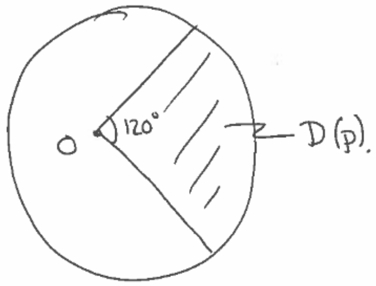
ILO5: assessed at medium level. Similar to examples done in lectures and in exercises. **ILO2**: assessed at a low level. Similar to exercises.

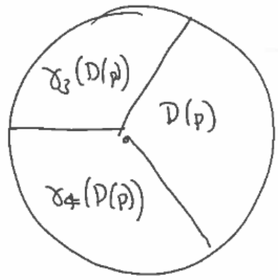
Solutions

(1) Choose $p \in \mathbb{D}$ such that $\gamma(p) \neq p$ for all $\gamma \in \Gamma \setminus \{\mathrm{id}\}$.
(2) For each $\gamma \in \Gamma \setminus \{\mathrm{id}\}$, construct the arc of geodesic $[p, \gamma(p)]$.
(3) Construct the perpendicular bisector $L_p(\gamma)$ of $[p, \gamma(p)]$.
(4) Let $H_p(\gamma)$ denote the half-plane of $\mathbb{D}$ determined by $L_p(\gamma)$ that contain p.
(5) Let $D(p) = \bigcap_{\gamma \in \Gamma \setminus \{\mathrm{id}\}} H_p(\gamma)$.

Note that the perpendicular bisector of $[p, \gamma_3(p)]$ is the diameter at angle 60 degrees, and similarly for γ_4. This allows us to determine $H_p(\gamma)$ for $\gamma = \gamma_3, \gamma_4$ and then $D(p)$, as illustrated. The tessellation of $\mathbb{D}$ is also illustrated.

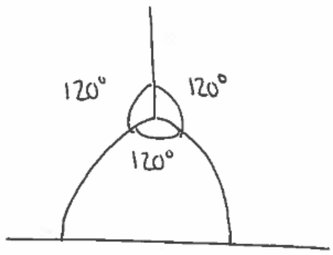






Let s_1 denote the side of $D(p)$ at angle -120 degrees; let s_2 denote the side of $D(p)$ at angle 120 degrees. Then s_1 is contained inside $L_p(\gamma_2)$ and so has side-pairing transformation $\gamma_{s_1} = \gamma_2^{-1}$, namely rotation through 120 degrees anticlockwise. Similarly, s_2 is contained inside $L_p(\gamma_1)$ and so has side-pairing transformation $\gamma_{s_2} = \gamma_1^{-1}$, namely rotation through 120 degrees clockwise.

The resulting tessellation of $\mathbb{H}$ is illustrated below



[14 marks]

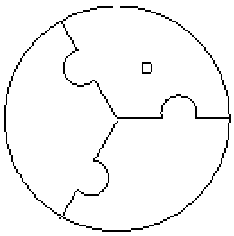
Feedback. This is very similar to an example given in the lectures. Many people missed out the part about calculating the side-pairing transformations. There were few good answers to the last part: drawing the picture in $\mathbb{H}$. Note that tessellation of $\mathbb{D}$ comprises: one point inside the hyperbolic plane with three geodesics emerging from that point (and with angle 120° between each geodesic) and going to the boundary.

Question Q2(iv)

Learning Outcome
ILO5: assessed at high level. Unseen.

Solutions
$D(p)$ in (iii) will be a rotation of $D(1/2)$

Any example along the lines of the one illustrated below will satisfy the requirements in the question.



[4 marks]

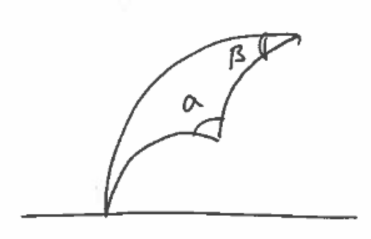
Feedback. Lots of people had the right idea as to what the question was asking, but drew the picture of a fundamental domain for the group of integer translations acting on $\mathbb{H}$, not the group above.

Question Q3(i)

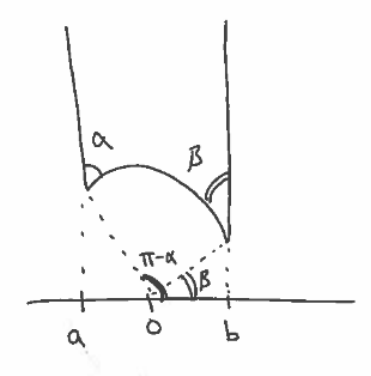
Learning Outcome
ILO3: assessed at a medium level.

This is bookwork, done in lectures.

Solutions
Suppose the triangle looks as in the picture.



Apply a Möbius transformation so that the vertex on the boundary is moved to ∞. Apply a translation and a dilation so that the semi-circular geodesic lies along the imaginary axis; these are both Möbius transformations. Möbius transformations are conformal and area-preserving, so this does not change the angles or area of Δ.



We have

$$\begin{aligned}
\text{Area}_{\mathbb{H}}(\Delta) &= \int\int_A \frac{dx\,dy}{y^2} \\
&= \int_{x=a}^{b} \int_{y=\sqrt{1-x^2}} \infty \frac{dy}{y^2}\, dx \\
&= \int_{x=a}^{b} \left. \frac{-1}{y} \right|_{\sqrt{1-x^2}}^{\infty} dx \\
&= \int_{x=a}^{b} \frac{dx}{\sqrt{1-x^2}} \\
&= \int_{\pi-\alpha}^{\beta} -1\, d\theta \\
&= \pi - (\alpha + \beta)
\end{aligned}$$

where we have used the substitution $x = \cos\theta$, noting that when $x = a$ we have $\theta = \pi - \alpha$ and when $y = b$ we have $\theta = \beta$.

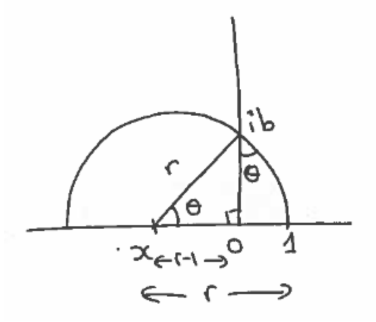
[8 marks]

Feedback. This is (part of) the proof of the Gauss-Bonnet Theorem.

Question Q3(ii)

Learning Outcome
ILO3: assessed at a medium level.

Solutions
Consider the picture below.



Similar to exercises.

By Pythagoras' Theorem, $(r-1)^2 + b^2 = r^2$. Expanding this out and cancelling the r^2 terms gives $r = (1+b^2)/2$.

Note that $\sin\theta = b/r$. Hence $\sin\theta = 2b/(1+b^2)$.

Consider the hyperbolic triangle with vertices at $(2+\sqrt{3})i$, 0, 1. The internal angles at 0,1 are 0. The internal angle at $(2+\sqrt{3})i$ corresponds to $a = 1, b = (2+\sqrt{3})$ in the above. Hence

$$\sin\theta = \frac{2(2+\sqrt{3})}{1+(4+4\sqrt{3}+3)} = \frac{2(2+\sqrt{3})}{4(2+\sqrt{3})} = \frac{1}{2}$$

so that $\theta = \pi/6$.

By the Gauss-Bonnet Theorem, the hyperbolic area of this triangle is $\pi - \pi/6 = 5\pi/6$.

[8 marks]

Feedback. There were a lot of mistakes in calculating $\sin\theta$. For some reason, many people made the algebraic slip of either writing

$$\sin\theta = \frac{2\sqrt{3}}{1+(4+4\sqrt{3}+3)}$$

rather than $2(2+\sqrt{3})$ in the numerator, or, when factorising the denominator, writing $1+(4+4\sqrt{3}+3) = 4(1+\sqrt{3})$ (or similar).

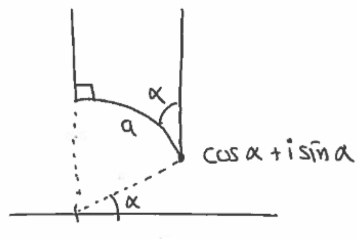
Question Q3(iii)

Learning Outcome

ILO3: assessed at a medium level. Bookwork, covered in lectures.

Solutions

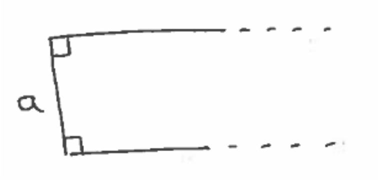
Apply a Möbius transformation so that the ideal vertex is at ∞ and the right-angle occurs at i, as illustrated.



As the finite side must lie along the unit circle, straightforward Euclidean geometry says that the other vertex is at $\cos\alpha + i\sin\alpha$ (see diagram). Then

$$\begin{aligned}
\cosh a &= \cosh d_{\mathbb{H}}(i, \cos\alpha + i\sin\alpha) \\
&= 1 + \frac{|\cos\alpha + i(\sin\alpha - 1)|^2}{2\sin\alpha} \\
&= 1 + \frac{\cos^2\alpha + (\sin\alpha - 1)^2|}{2\sin\alpha} \\
&= \frac{2\sin\alpha + \cos^2\alpha + \sin^2\alpha - 2\sin\alpha + 1}{2\sin\alpha} \\
&= \frac{1}{\sin\alpha}.
\end{aligned}$$

A Euclidean analogue would involve a degenerate 'triangle' with one vertex at infinity (and so two parallel sides). In this case. the length of the finite side can be arbitrary (see picture).



[8 marks]

Feedback. The angle of parallelism was proved in the lectures. The angle of parallelism concerns a right-angled triangle with one vertex at infinity. There are several ways of trying to come up with a Euclidean example (one of which is above); none of them make any real sense. Any sensible and reasoned answer was accepted (but you had to give a reason: just writing 'yes' or 'no' isn't sufficient).

Question Q3(iv)

Learning Outcome
ILO3: assessed at high level. Unseen, involves putting together ideas from the course in a way that students haven't considered.

Solutions
By the angle of parallelism formula, we can find the angle α.

We have

$$\begin{aligned}\cosh(\log(2+\sqrt{3})) &= \frac{e^{\log(2+\sqrt{3})}+e^{-(\log(2+\sqrt{3}))}}{2} \\ &= \frac{2+\sqrt{3}+\frac{1}{2+\sqrt{3}}}{2} \\ &= \frac{(2+\sqrt{3})(2+\sqrt{3})+1}{2(2+\sqrt{3})} \\ &= \frac{4+4\sqrt{3}+3+1}{2(2+\sqrt{3})} \\ &= \frac{4(2+\sqrt{3})}{2(2+\sqrt{3})}=2.\end{aligned}$$

Hence $\sin\alpha = 1/2$ so that $\alpha = \pi/6$.

By the Gauss-Bonnet Theorem, the hyperbolic area of this triangle is $\pi - (\pi/2+\pi/6) = \pi/3$.

[6 marks]

Feedback. There were lots of right answers here. Note the side of finite hyperbolic length has length $\log(2+\sqrt{3})$—a rather odd choice of number! This suggests that this isn't a randomly picked number and that something will simplify quite considerably later on. If you end up with $\sin\alpha$ being equal to something that does not allow you to write down θ explicitly then you've probably gone wrong.

Question Q4(a)(i)

Learning Outcome
ILO7: assessed at low level. Standard definitions from the course.

Solutions
An elliptic cycle $\mathcal{E}$ satisfies the *elliptic cycle condition* if there exists an integer $m \geq 1$ such that $m \times \mathrm{sum}\mathcal{E} = 2\pi$ (here $\mathrm{sum}\mathcal{E}$ denotes the angle sum along the elliptic cycle).

A parabolic cycle $\mathcal{P}$ satisfies the *parabolic cycle condition* if the corresponding parabolic cycle transformation is either parabolic or the identity.

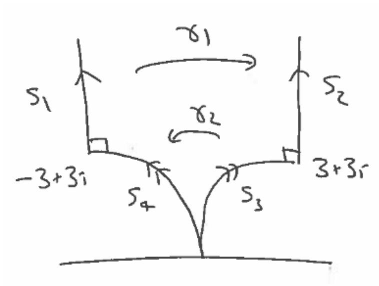
[4 marks]

Feedback. Standard definitions!

Question Q4(a)(ii)

Learning Outcome
ILO7: assessed at medium level. Similar to example sheets. **ILO8**: assessed at medium level. Description of construction of $\mathbb{H}/\Gamma$ is similar to exercise sheets.

Solutions
Label the diagram as below



We have the elliptic cycle

$$\begin{pmatrix} -3+3i \\ s_1 \end{pmatrix} \overset{\gamma_1}{\to} \begin{pmatrix} 3+3i \\ s_2 \end{pmatrix} \overset{*}{\to} \begin{pmatrix} 3+3i \\ s_3 \end{pmatrix} \overset{\gamma_2}{\to} \begin{pmatrix} -3+3i \\ s_4 \end{pmatrix} \overset{*}{\to} \begin{pmatrix} -3+3i \\ s_1 \end{pmatrix}$$

This gives
- elliptic cycle $\mathcal{E} = -3+3i \to 3+3i$,
- elliptic cycle transformation $\gamma_2\gamma_1$,
- $\operatorname{sum}\mathcal{E} = \pi/2 + \pi/2 = \pi$.

Hence the elliptic cycle condition holds with $m = 2$.

We have the parabolic cycle

$$\begin{pmatrix} 0 \\ s_3 \end{pmatrix} \overset{\gamma_2}{\to} \begin{pmatrix} 0 \\ s_4 \end{pmatrix} \overset{*}{\to} \begin{pmatrix} 0 \\ s_3 \end{pmatrix}.$$

This gives
- parabolic cycle $\mathcal{P}_1 = 0$,
- parabolic cycle transformation γ_2,

Hence the parabolic cycle condition will hold if γ_2 is parabolic or the identity. Note that

$$\frac{-3z}{z-3} = z \Leftrightarrow -3z = z^2 - 3z \Leftrightarrow z = 0$$

so that γ_2 is parabolic.

We have the parabolic cycle

$$\begin{pmatrix} \infty \\ s_1 \end{pmatrix} \overset{\gamma_1}{\to} \begin{pmatrix} \infty \\ s_2 \end{pmatrix} \overset{*}{\to} \begin{pmatrix} \infty \\ s_1 \end{pmatrix}.$$

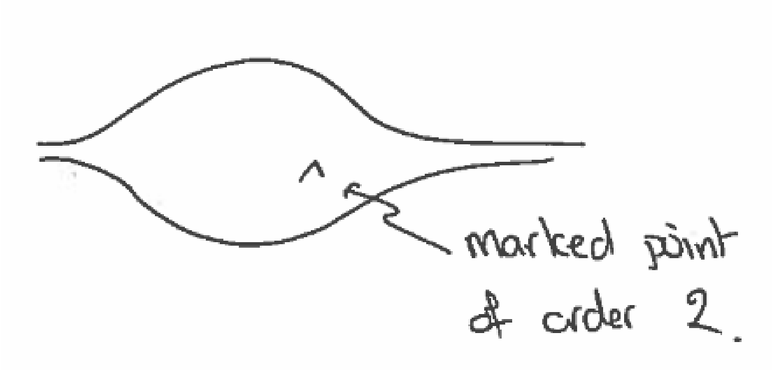
This gives
- parabolic cycle $\mathcal{P}_2 = \infty$,
- parabolic cycle transformation γ_1,

Hence the parabolic cycle condition will hold if γ_2 is parabolic or the identity. Note that γ_2 is a translation, so that γ_2 is parabolic.

Denote γ_1, γ_2 by abstract symbols a, b, resptively. By Poincaré's Theorem, γ_1, γ_2 generate a Fuchsian group Γ with presentation

$$\langle a, b \mid (ba)^2 = e \rangle.$$

The quotient space $\mathbb{H}/\Gamma$ has genus 0, two cusps and one marked point of order 2.



To check that γ_1, γ_2 satisfy the relation $(ba)^2 = e$, we can consider the matrix product

$$\left(\begin{bmatrix} -3 & 0 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} \right)^2 = \begin{bmatrix} -3 & -18 \\ 1 & 3 \end{bmatrix}^2 = \begin{bmatrix} -9 & 0 \\ 0 & -9 \end{bmatrix}$$

which corresponds to the identity Möbius transformation, as expected.

[14 marks]

Feedback. Calculating elliptic cycles, parabolic cycles, checking the ECC and PCC, etc, are (as I hope you've realised) usually pretty easy. Almost everybody who checked the PCC by calculating the trace remembered to normalise, and I count this amongst my greatest achievements this semester!!!
There was some confusion over how to draw $\mathbb{H}/\Gamma$. I think this is easiest seen by thinking in your head how to glue together the paired sides as indicated. First note that there are two parabolic cycles, hence two cusps. Imagine the quadrilateral as a diamond-shaped sheet of paper (or piece of pastry), albeit with two opposing corners at infinity. Glue the bottom left side to the bottom right side, and the top left side to the top right side. This gives a surface of genus 0 (no holes in it) and two cusps (with the pastry analogy: it looks like a Cornish pasty that goes off to infinity at both ends).

Question Q4(b)(i)

Learning Outcome
ILO6: assessed at low level. Bookwork: standard definition from the course.

Solutions
$\mathcal{F}_k$ contains the set of all finite words $a_{i_1}a_{i_2}\ldots a_{i_n}$ in symbols a_{i_j} chosen from $\mathcal{S} \cup \mathcal{S}^{-1}$ subject to $a_{i_{j\pm1}} \neq a_{i_j}^{-1}$.

The group operation is concatenation of words.
The group identity is the empty word e.
If $w = a_{i_1}a_{i_2}\ldots a_{i_n}$ then $w^{-1} = a_{i_n}^{-1}\ldots a_{i_2}^{-1}a_{i_1}^{-1}$.

[4 marks]

Feedback. Read the question: it tells you what you need to define! Many people missed out either how to do inverses or what the group operation actually is, etc, but got the remainder right. This suggests that you know what the answer is, but you're not writing it down.

Question Q4(b)(ii)

Learning Outcome
ILO6: assessed at medium (first part) high (second part) level. The

Solutions
In $\mathcal{F}_2$ with symbols a, b:

- words of length 1: a, b, a^{-1}, b^{-1}.

first part is similar to exercise sheets. The second part is unseen.

- words of length 2: $a^2, ab, ab^{-1}, ba, ba^{-1}, b^2, a^{-2}, a^{-1}b, a^{-1}b^{-1}, b^{-1}a, b^{-1}a^{-1}, b^{-2}$.

There are $4 \times 3^{n-1}$ words of length. This is easily proved by induction. The base case is above. Given a word of length $n-1$ one can form a new word of length n by concatenating any one of 3 symbols at the end (not 4, as otherwise it would cancel with the final symbol of the original word). As there are no other relations in $\mathcal{F}_2$, all these words are distinct.

[8 marks]

Feedback. The first part was generally answered correctly. For the second part, many people had the right idea but got the wrong formula (involving factorials or n-choose-r, etc).