2 hours

THE UNIVERSITY OF MANCHESTER

HYPERBOLIC GEOMETRY

?? Jan 2020

??:?? – ??:??

Answer **THREE** of the **FOUR** questions.
If all four questions are attempted then credit will be given for the three best answers.

Electronic calculators may be used in accordance with the University regulations

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Notation: Throughout, $\mathbb{H}$ denotes the upper half-plane, $\partial\mathbb{H}$ denotes the boundary of $\mathbb{H}$, $\mathbb{D}$ denotes the Poincaré disc, and $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$. All logarithms are natural logarithms.

1.

(i) Recall that the (Euclidean) circle in $\mathbb{C}$ with centre $z_0 \in \mathbb{C}$ and radius $r > 0$ is given by the equation $|z - z_0|^2 = r^2$. Show that this equation can be written in the form

$$z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0, \quad \beta \in \mathbb{C}, \gamma \in \mathbb{R}$$

and determine β, γ in terms of z_0, r.

Show that the equation of a (Euclidean) circle in $\mathbb{C}$ with real centre z_0 has the form

$$z\bar{z} + \beta z + \beta\bar{z} + \gamma = 0, \quad \beta, \gamma \in \mathbb{R} \tag{1}$$

[6 marks]

(ii) Consider the points $-5 + 12i$, $12 + 5i \in \mathbb{H}$. Find the equation of the geodesic (i.e. find an equation of the form (1)) that passes through these two points.

This geodesic is a semi-circle. Determine its centre z_0 and radius r. Hence write down the end-points of this geodesic.

Write down a Möbius transformation of $\mathbb{H}$ that maps this geodesic to the imaginary axis.

The point $z_1 = \dfrac{39}{5} + \dfrac{52}{5}i$ also lies on this geodesic. Briefly explain how you would construct a Möbius transformation that maps this geodesic to the imaginary axis and maps the point z_1 to i. (You do not need explicitly calculate this Möbius transformation; instead, your answer should explain how you would do it.)

[8 marks]

(iii) Let $0 < a < b$. Let σ be a path from ia to ib. Prove that $\text{length}_{\mathbb{H}}(\sigma) \geq \log b/a$ with equality if, and only if, σ is the straight line along the imaginary axis from ia to ib.

[8 marks]

(iv) Consider the following statement.

> Let H_1, H_2 be two geodesics in $\mathbb{H}$ that do not intersect. Then there exists a unique geodesic in $\mathbb{H}$ that passes through both H_1 and H_2 at right-angles.

Suppose that H_1, H_2 have *distinct* end-points on $\partial\mathbb{H}$. Prove that, in this case, the above statement is true. (Hint: Without loss of generality you can assume that H_1 is the imaginary axis. What geodesics pass through H_1 at right-angles?)

Is the Euclidean analogue of the above statement true?

[6 marks]

2.

(i) Recall that a Möbius transformation of $\mathbb{D}$ is a transformation of the form

$$\gamma(z) = \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}$$

where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 - |\beta|^2 > 0$.

In each of the following cases, state whether the transformation is a Möbius transformation of $\mathbb{D}$ or not giving a brief reason for your answer:

$$\text{(a) } \gamma(z) = e^{i\theta} z,\ \theta \in \mathbb{R}, \qquad \text{(b) } \gamma(z) = \frac{-1}{z}.$$

[4 marks]

(ii) Let

$$\gamma_1(z) = \frac{\alpha_1 z + \beta_1}{\bar{\beta}_1 z + \bar{\alpha}_1}, \quad \gamma_2(z) = \frac{\alpha_2 z + \beta_2}{\bar{\beta}_2 z + \bar{\alpha}_2} \in \text{Möb}(\mathbb{D})$$

be two Möbius transformations of the Poincaré disc $\mathbb{D}$. (Here $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$ and $|\alpha_1|^2 - |\beta_1|^2 > 0, |\alpha_2|^2 - |\beta_2|^2 > 0$.)

Show that the composition $\gamma_1 \gamma_2$ is a Möbius transformation of $\mathbb{D}$.

[8 marks]

(iii) Let $\Gamma \subset \text{Möb}(\mathbb{D})$ be a Fuchsian group. Briefly outline a procedure which will generate a Dirichlet region for Γ.

Let γ_3 denote rotation around the origin through 120 degrees anticlockwise; let γ_4 denote rotation around the origin 120 degrees clockwise. Let $\Gamma = \{\text{id}, \gamma_3, \gamma_4\}$.

Let $p = 1/2$ and determine the Dirichlet polygon $D(p)$.

Sketch the resulting tessellation in $\mathbb{D}$.

Sketch the corresponding tessellation in the upper half-plane $\mathbb{H}$.

[12 marks]

(iv) Give an example of a convex hyperbolic polygon D which cannot be a Dirichlet polygon for any Fuchsian group. Justify your answer.

[6 marks]

3. Throughout this question you may use the fact that $\cosh d_{\mathbb{H}}(z,w) = 1 + \frac{|z-w|^2}{2\,\mathrm{Im}(z)\,\mathrm{Im}(w)}$. You may also use the fact that $\sin \pi/6 = 1/2$.

(i) Recall that if $A \subset \mathbb{H}$ then $\mathrm{Area}_{\mathbb{H}}(A) = \int\int_A \frac{dx\,dy}{y^2}$.

Let Δ be a hyperbolic triangle with internal angles $\alpha, \beta, 0$. Prove that $\mathrm{Area}_{\mathbb{H}}(A) = \pi - (\alpha + \beta)$. (You may *NOT* assume the Gauss-Bonnet theorem; you should prove this directly from the formula for the hyperbolic area above.)

[8 marks]

(ii) Consider the diagram in Figure 1(i) below. Show that

$$\sin\theta = \frac{2b}{1+b^2}. \qquad (2)$$

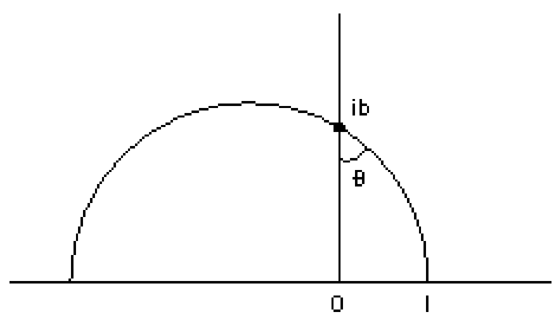

Figure 1: See Q3(ii).

[Hint: suppose the geodesic through 1 and ib is a semi-circle with centre x and radius r. Consider the (Euclidean) right-angled triangle with vertices at x, 0, ib and use the (Euclidean) Pythagoras Theorem.]

Consider the hyperbolic triangle in $\mathbb{H}$ with vertices at $(2+\sqrt{3})i$, 0, 1 as illustrated in Figure 1(ii). Use the Gauss-Bonnet Theorem and (2) to calculate the hyperbolic area of this triangle.

[8 marks]

(iii) Let Δ be a right-angled hyperbolic triangle with one ideal vertex and internal angles $\alpha, 0, \pi/2$. Then Δ has one side with finite hyperbolic length; let a denote the hyperbolic length of this side.

Prove the angle of parallelism formula: $\sin\alpha = 1/\cosh a$.

Is there a Euclidean analogue of this result?

[8 marks]

(iv) Let Δ be a right-angled hyperbolic triangle with one ideal vertex. Suppose that the side of Δ of finite hyperbolic length has hyperbolic length $\log(2+\sqrt{3})$.

Calculate $\text{Area}_{\mathbb{H}}(\Delta)$.

[6 marks]

4.

(a) (i) Let $\mathcal{E}$ be an elliptic cycle with corresponding elliptic cycle transformation γ. What does it mean to say that $\mathcal{E}$ satisfies the *elliptic cycle condition*?

Let $\mathcal{P}$ be a parabolic cycle with corresponding parabolic cycle transformation γ. What does it mean to say that $\mathcal{P}$ satisfies the *parabolic cycle condition*?

[4 marks]

(ii) Consider the hyperbolic polygon as illustrated in Figure XXXX.

[picture]

Define

$$\gamma_1(z) = z + 6, \quad \gamma_2(z) = \frac{-3z}{z-3}. \tag{3}$$

Use Poincaré's Theorem to show that γ_1 and γ_2 generate a Fuchsian group Γ. Give a presentation of Γ in terms of generators and relations. Briefly describe the quotient space $\mathbb{H}/\Gamma$.

Show by explicit calculation that γ_1, γ_2, as defined in (3), satisfy the relation or relations that you have given in your presentation of Γ.

[14 marks]

(b) (i) Let $\mathcal{S} = \{a_1, \ldots, a_k\}$ be a finite set of symbols. Briefly explain how to construct the free group on k generators, $\mathcal{F}_k$.

(Your answer should include: a description of the elements of $\mathcal{F}_k$, a description of the group operation, a description of the group identity, a description of how to find the inverse of an element in $\mathcal{F}_k$. You do not need to prove that the group operation is well-defined.)

[4 marks]

(ii) Consider $\mathcal{F}_2$, the free group on 2 generators a, b. Show that there are 4 words of length 1 and 12 words of length 2 in $\mathcal{F}_2$.

How many words of length n are there in $\mathcal{F}_2$? Justify your answer.

[8 marks]

END OF EXAMINATION PAPER