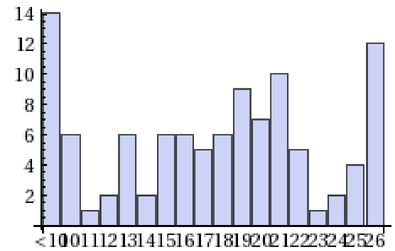
Solutions for MATH32051 Hyperbolic Geometry Coursework Solutions, October 2019

The distribution of marks is shown below.



Question Q1

Learning Outcome
ILO3: prove results (Gauss-Bonnet Theorem, angle formulae for triangles, etc, as listed in the syllabus) in hyperbolic trig and use them to calculate angles, side lengths, hyperbolic areas, etc, of hyperbolic triangles and polygons.

Solutions

(i) Let Δ be a hyperbolic triangle with internal angles α, β, γ. Then $\text{Area}_{\mathbb{H}}(\Delta) = \pi - (\alpha + \beta + \gamma)$.

[2 marks]

Feedback: Almost everybody got this right.

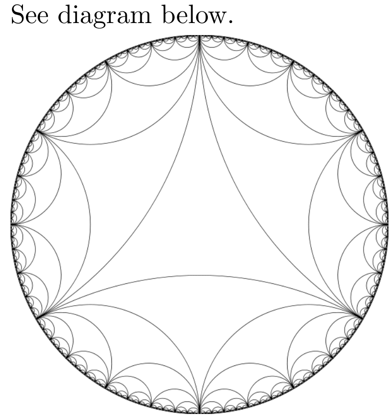
(ii) Let α denote the internal angle. There are eight triangles meeting at each vertex. Hence $8\alpha = 2\pi$, so that $\alpha = \pi/4$.
By the Gauss-Bonnet Theorem,

$$\text{Area}_{\mathbb{H}}(\Delta) = \pi - (\alpha + \alpha + \alpha) = \pi - \frac{3\pi}{4} = \frac{\pi}{4}.$$

[4 marks]

Feedback: Some people miscounted the number of triangles meeting at each vertex.

(iii) See diagram below.



Feedback: The triangles are all ideal, so there cannot be any vertices in $\mathbb{D}$ itself: they must all be on the boundary. Many of your diagrams were less-than-convincing that the geodesics met the boundary at right angles; I gave you the benefit of the doubt as it is quite hard to draw!

[4 marks]

Question Q2

Learning Outcome
ILO4: classify Möbius transformations in terms of their actions on the hyperbolic plane.

Solutions

(i) Let $\gamma(z) = (az+b)/(cz+d)$, $a, b, c, d \in \mathbb{R}$ and assume that this is in normalised form so that $ad - bc = 1$. Then $\tau(\gamma) := (a+d)^2$.
- if $\tau(\gamma) > 4$ then γ is hyperbolic
- if $\tau(\gamma) = 4$ then γ is parabolic
- if $\tau(\gamma) \in [0, 4)$ then γ is elliptic.

[4 marks]

Feedback: Many of you forgot to include the requirement that γ needs to be written in normalised form. A few of you wrote that ‘if

$\tau(\gamma) \in (0,4)$ then γ is elliptic' (or similar). Note that $\tau(\gamma)$ can equal 0. For example, $\gamma(z) = -1/z$ has $\tau(\gamma) = 0$ and is elliptic (it has a unique fixed point at i).

(ii) Here '$ad - bc$' $= 2 \times (-3) - (-3) \times 5 = -6 + 15 = 9$. Hence we can write γ in normalised form as

$$\gamma(z) = \frac{\frac{2}{3}z - \frac{3}{3}}{\frac{5}{3}z - \frac{3}{3}}.$$

Hence $\tau(\gamma) = \left(\frac{2}{3} - \frac{3}{3}\right)^2 = (-1/3)^2 = 1/9 \in [0,4)$. Hence γ is elliptic.
We have that

$$\gamma(z_0) = z_0 \Leftrightarrow 5z_0^2 - 5z_0 + 3 = 0 \Leftrightarrow z_0 = \frac{25 + i\sqrt{35}}{10}$$

(for solutions in $\mathbb{H}$). Hence γ has one fixed point in $\mathbb{H}$ and none in $\partial\mathbb{H}$. [4 marks]

Feedback: If a kitten died every time somebody forgot to normalise a Möbius transformation then there would be 28 fewer kittens in the world today. Please remember to normalise before calculating the trace.
Many of you lost easy marks through poor exam technique. The question asks you to check your answer by explicitly calculating the fixed points. This is easy: it's just solving a quadratic equation. So if you don't do it then you're throwing marks away.

(iii) Let $\gamma_1(z) = (az + b)/(cz + d)$. As $\gamma_1(0) = 0$ we have $b = 0$. As $\gamma_1(\infty) = \infty$ we have $c = 0$. Hence $\gamma_1(z) = (a/d)z$, a dilation. [2 marks]
Suppose that γ has fixed points at $a < b$. Let $g(z) = (z - b)/(z - a)$. Then g is a Möbius transformation of $\mathbb{H}$ and maps a, b to $0, \infty$. Consider $\gamma_1 = g\gamma g^{-1}$. Then γ_1 is conjugate to γ (as $\gamma = g^{-1}\gamma_1 g$).
Moreover, $\gamma_1(z_0) = z_0$ iff $g\gamma g^{-1}(z_0) = z_0$ iff $\gamma(g^{-1}(z_0)) = g^{-1}(z_0)$ iff $g^{-1}(z_0) = a, b$ iff $z_0 = g(a), g(b)$ iff $0, \infty$. By the result in the first paragraph, γ_1 is a dilation. [6 marks]

Feedback: The second part of this was by far the worst answered question on the test. Many of you tried to start with a dilation and then show that this was conjugate to γ (this is possible, but *extremely* messy and nobody managed to do it correctly. This is a proof that we saw (in fact, saw three times as there are very similar proofs in the parabolic and elliptic cases) in the lectures.