2 hours

THE UNIVERSITY OF MANCHESTER

HYPERBOLIC GEOMETRY

15 January 2018

09:45 – 11:45

Answer **all** four questions in **Section A** (40 marks in all) and
and **two** of the three questions in **Section B** (30 marks each).
If all three questions from Section B are attempted then credit will be given for the two best answers.

University approved calculators may be used.

Notation: Throughout, $\mathbb{H}$ denotes the upper half-plane, $\partial\mathbb{H}$ denotes the boundary of $\mathbb{H}$, $\mathbb{D}$ denotes the Poincaré disc, and $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$.

SECTION A

Answer **ALL** four questions

A1.

(i) Recall that the equation of a straight line in $\mathbb{R}^2$ is given by $ax + by + c = 0$ where $a, b, c \in \mathbb{R}$. Show that the equation of a straight line in $\mathbb{C}$ can be written as $\beta z + \bar{\beta}\bar{z} + \gamma = 0$ where $\beta \in \mathbb{C}$ and $\gamma \in \mathbb{R}$.

[4 marks]

(ii) Which values of $\beta \in \mathbb{C}$ correspond to equations of horizontal straight lines?

[2 marks]

(iii) Find an equation (i.e. determine β and γ) of the form $\beta z + \bar{\beta}\bar{z} + \gamma = 0$ that describes the straight line through $-1 + 2i$ and $3 + 2i$.

[4 marks]

A2.

(i) Let $\gamma_1, \gamma_2 \in \text{Möb}(\mathbb{H})$. What does it mean to say that γ_1 and γ_2 are conjugate Möbius transformations?

[2 marks]

(ii) Recall that a Möbius transformation of $\mathbb{H}$ is said to be hyperbolic if it has two fixed points on $\partial\mathbb{H}$ and no fixed points in $\mathbb{H}$.

Let $k > 0$. Show that the dilation $\gamma(z) = kz$ is hyperbolic.

[2 marks]

(iii) Let $\gamma \in \text{Möb}(\mathbb{H})$ be hyperbolic. Show that γ is conjugate to a dilation.

[You may use without proof the facts that, (i) given a geodesic in $\mathbb{H}$, there exists a Möbius transformation of $\mathbb{H}$ that maps that geodesic to the imaginary axis, (ii) if a Möbius transformation has fixed points at 0 and ∞ then it is a dilation or the identity.]

[8 marks]

A3. Let Γ be a Fuchsian group acting on $\mathbb{D}$.

(i) Let F be an open subset of $\mathbb{D}$. What does it mean to say that F is a fundamental domain for Γ?

[2 marks]

(ii) Let

$$\Gamma = \{\gamma_k \mid \gamma_k(z) = e^{2\pi i k/5} z,\ k = 0, 1, 2, 3, 4\}$$

denote the Fuchsian group generated by a rotation through angle $2\pi/5$. Show (by drawing the associated tessellation of $\mathbb{D}$) that $F = \{z \in \mathbb{D} \mid -\pi/5 < \arg(z) < \pi/5\}$ is a fundamental domain for Γ.

[4 marks]

A4.

(i) What does it mean to say that $\Gamma \subset \text{Möb}(\mathbb{H})$ is a Fuchsian group?

What is meant by the orbit $\Gamma(z)$ of $z \in \mathbb{H} \cup \partial\mathbb{H}$?

[4 marks]

(ii) Let $\Gamma = \{\gamma_n \mid \gamma_n(z) = 2^n z,\ n \in \mathbb{Z}\}$. Sketch the orbit $\Gamma(1+i)$ of $1+i$.

[2 marks]

(iii) It was stated in the course that Γ is a Fuchsian group if and only if, for all $z \in \mathbb{H}$, the orbit $\Gamma(z)$ of z is a discrete subset of $\mathbb{H}$.

Let $\Gamma = \text{PSl}(2, \mathbb{Z})$. Find a point $z \in \partial\mathbb{H}$ for which $\Gamma(z)$ is not a discrete subset of $\partial\mathbb{H}$. Explain briefly why, in the example you give, $\Gamma(z)$ is not discrete.

[6 marks]

SECTION B

Answer **TWO** of the three questions

B5. Let $X \subset \mathbb{C}$ and let $\rho : X \to \mathbb{R}$ be a continuous positive function. Let $\sigma : [a,b] \to X$ be a path in X and define

$$\text{length}_\rho(\sigma) = \int_\sigma \rho = \int_a^b \rho(\sigma(t))|\sigma'(t)|\,dt.$$

We can then define the metric d_ρ on X by

$$d_\rho(z,w) = \inf\left\{\text{length}_\rho(\sigma) \mid \sigma \text{ is a piecewise differentiable path from } z \text{ to } w\right\}.$$

A map $\gamma : X \to X$ is said to be an *isometry* if $d_\rho(\gamma(z), \gamma(w)) = d_\rho(z,w)$ for all $z, w \in X$.

(i) Show that d_ρ satisfies the triangle inequality, i.e. $d_\rho(z_1, z_3) \leq d_\rho(z_1, z_2) + d_\rho(z_2, z_3)$ for all $z_1, z_2, z_3 \in X$.

[8 marks]

(ii) Suppose that $\gamma : X \to X$ is differentiable. Prove that

$$\text{length}_\rho(\gamma \circ \sigma) = \text{length}_\rho(\sigma) \text{ for all piecewise differentiable paths } \sigma \qquad (*)$$

if and only if $\rho(\gamma(z))|\gamma'(z)| = \rho(z)$ for all $z \in X$.

Briefly explain why, if (*) holds, then γ is an isometry.

[You may use, without proof, the fact that if f is continuous and $\int_\sigma f = 0$ for all piecewise smooth paths σ then $f = 0$.]

[8 marks]

(iii) Consider the case when $X = \mathbb{H}$ and $\rho(z) = 1/\text{Im}z$.

Let $\gamma(z) = (az+b)/(cz+d)$, $a,b,c,d \in \mathbb{R}$, $ad - bc > 0$, be a Möbius transformation of $\mathbb{H}$. It was proved in the course that

$$\text{Im}\gamma(z) = \frac{ad-bc}{|cz+d|^2}\text{Im}z, \text{ and } \gamma'(z) = \frac{ad-bc}{(cz+d)^2}.$$

Use (ii) and these facts to show that Möbius transformations of $\mathbb{H}$ are isometries.

[2 marks]

(iv) Suppose that $\rho : \mathbb{H} \to \mathbb{R}$ is such that $d_\rho(\gamma(z), \gamma(w)) = d_\rho(z,w)$ for all $z, w \in \mathbb{H}$ and all $\gamma \in \text{Möb}(\mathbb{H})$.

By considering translations, $\gamma(z) = z + b$, use the result in (ii) to show that $\rho(x + iy) = \rho(y)$, i.e. $\rho(z)$ depends only on the imaginary part of z.

By considering dilations, $\gamma(z) = kz$, now use (ii) again to show that $\rho(z)$ is a constant multiple of $1/\text{Im}z$.

[8 marks]

(v) Now take $X = \mathbb{C}$. Let $\rho : \mathbb{C} \to \mathbb{R}$ be a positive continuous function so that d_ρ defines a metric on $\mathbb{C}$. Let $a + ib \in \mathbb{C}$ and define the translation $T_{a,b}(x + iy) = (x + a) + i(y + b)$.

Suppose that every translation $T_{a,b}$ is an isometry with respect to d_ρ. Show that d_ρ is a constant multiple of the Euclidean metric on $\mathbb{C}$.

[4 marks]

B6. Recall that the hyperbolic area in the upper half-plane $\mathbb{H}$ of a subset $A \subset \mathbb{H}$ is given by

$$\text{Area}_{\mathbb{H}}(A) = \int\int_A \frac{1}{y^2}\, dx\, dy.$$

(i) Let Δ be a hyperbolic triangle in $\mathbb{H}$ with internal angles α, β, γ.

Prove the Gauss-Bonnet theorem:

$$\text{Area}_{\mathbb{H}}(\Delta) = \pi - (\alpha + \beta + \gamma).$$

[If you reduce the triangle Δ to a special case, then you should briefly justify how and why this is valid.]

[10 marks]

(ii) Let Q be a hyperbolic quadrilateral with internal angles $\alpha, \beta, \gamma, \delta$. Show that

$$\text{Area}_{\mathbb{H}}(\mathrm{Q}) = 2\pi - (\alpha + \beta + \gamma + \delta). \tag{1}$$

[4 marks]

(iii) The area in the Poincaré disc model $\mathbb{D}$ of a subset $A \subset \mathbb{D}$ is given by

$$\text{Area}_{\mathbb{D}}(A) = \int\int_A \frac{4}{(1-(x^2+y^2))^2}\, dx\, dy.$$

Let $r \in (0, 1)$. Let $C_r = \{z \in \mathbb{D} \mid |z| \leq r\}$. Prove that

$$\text{Area}_{\mathbb{D}}(C_r) = \frac{4\pi r^2}{1 - r^2}.$$

[You may use the fact that the area form in polar co-ordinates is $\rho\, d\rho\, d\theta$.]

[4 marks]

(iv) Recall that a regular hyperbolic n-gon has n sides, each of equal length, and n equal internal angles.

Show (by explicit construction in $\mathbb{D}$, considering a 4-gon with vertices at $r, ir, -r, -ir$ and applying parts (ii), (iii) above) that there exists a regular hyperbolic 4-gon with internal angle α if and only if $\alpha \in [0, \pi/2)$.

[10 marks]

(v) Show that if there is a tessellation of the hyperbolic plane by regular hyperbolic 4-gons with k polygons meeting at each vertex then $k > 4$.

[2 marks]

B7.

(i) Let $\gamma(z) = (az+b)/(cz+d) \in \text{Möb}(\mathbb{H})$, $a,b,c,d \in \mathbb{R}$, $ad-bc>0$. Recall that γ is parabolic if it has one fixed point on $\partial\mathbb{H}$ and no fixed points in $\mathbb{H}$.

Assume that $c \neq 0$. By considering the equation $\gamma(z_0) = z_0$, show that γ has a unique fixed point on $\partial\mathbb{H}$ (and so is parabolic) if and only if $(d-a)^2 + 4bc = 0$.

[2 marks]

(ii) Let $k>0$ and let $\ell > 1$. Define

$$\gamma_1(z) = \frac{kz}{(k+1)z+1}, \quad \gamma_2(z) = \frac{(1-\ell^2)z - 2\ell^2}{2z + (1-\ell^2)}.$$

Show that γ_1 pairs the sides s_1 and s_2 of the quadrilaterial with vertices at $-1, 0, 1, i\ell$, as illustrated in Figure 1 below. (One can also show that γ_2 pairs the sides s_3 and s_4 as illustrated; you do not need to do this.)

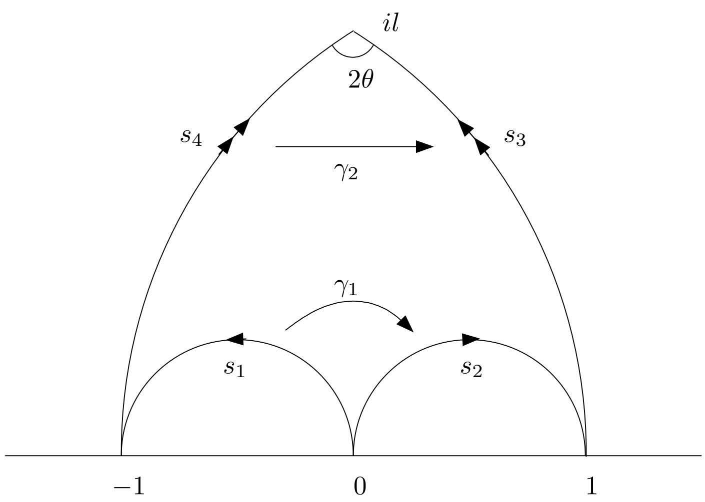

Figure 1: A hyperbolic quadrilateral, see B7(ii).

[2 marks]

(iii) Let $\mathcal{E}$ be an elliptic cycle. What does it mean to say that $\mathcal{E}$ satisfies the Elliptic Cycle Condition?

Let $\mathcal{P}$ be a parabolic cycle. What does it mean to say that $\mathcal{P}$ satisfies the Parabolic Cycle Condition?

[4 marks]

(iv) Show that, in the diagram given in Figure 1, there is one elliptic cycle and two parabolic cycles.

Use Poincaré's Theorem to determine conditions on k and θ for which γ_1 and γ_2 generated a Fuchsian group. When do γ_1, γ_2 generate a Fuchsian group Γ, give a presentation of Γ in terms of generators and relations.

[12 marks]

(v) Consider the diagram in Figure 2 below. Show that

$$\sin\theta = \frac{2\ell}{\ell^2+1}.$$

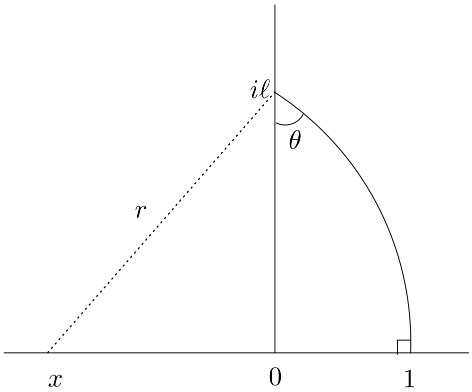

Figure 2: See B7(v).

Hence give explicit transformations γ_1, γ_2 that generate a group with presentation $\langle a, b \mid b^6 = e\rangle$. [Hint: suppose the geodesic through 1 and $i\ell$ is a semi-circle with centre x and radius r and consider the (Euclidean) right-angled triangle with vertices $x, 0, i\ell$, using the (Euclidean) Pythagoras theorem to find a relationship between ℓ and r. You may also assume that $\sin\pi/6 = 1/2$.]

[10 marks]

END OF EXAMINATION PAPER