25. Solutions

Solution 1.1
We write $T_{\theta,a}(x, y)$ in the form $R_\theta(x, y) + (a_1, a_2)$ where R_θ denotes the 2×2 matrix that rotates the plane about the origin by angle θ.

(i) (a) Let $T_{\theta,a}, T_{\theta',a'} \in G$. We have to show that the composition $T_{\theta,a}T_{\theta',a'} \in G$. Now

$$\begin{aligned}
T_{\theta,a}T_{\theta',a'}(x,y) &= T_{\theta,a}(T_{\theta',a'}(x,y)) \\
&= T_{\theta,a}(R_{\theta'}(x,y) + (a'_1, a'_2)) \\
&= R_\theta(R_{\theta'}(x,y) + (a'_1, a'_2)) + (a_1, a_2) \\
&= R_\theta R_{\theta'}(x,y) + (R_\theta(a'_1, a'_2) + (a_1, a_2)) \\
&= T_{\theta+\theta', R_\theta(a'_1,a'_2)+(a_1,a_2)}(x,y)
\end{aligned}$$

where we have used the observation that $R_\theta R_{\theta'} = R_{\theta+\theta'}$. As $T_{\theta+\theta', R_\theta(a'_1,a'_2)+(a_1,a_2)} \in G$, the composition of two elements of G is another element of G, hence the group operation is well-defined.

(b) This is trivial: composition of functions is already known to be associative.

(c) The identity map on $\mathbb{R}^2$ is the map that leaves every point alone. We choose $\theta = 0$ and $a = (0, 0)$.

$$T_{0,(0,0)}(x, y) = R_0(x, y) + (0, 0).$$

As R_0 is the rotation through angle 0, it is clearly the identity matrix, so that $R_0(x, y) = (x, y)$. Hence $T_{0,(0,0)}(x, y) = (x, y)$. Hence G has an identity element.

(d) Let $T_{\theta,a} \in G$. We want to find an inverse for $T_{\theta,a}$ and show that it lies in G. Write

$$T_{\theta,a}(x, y) = (u, v).$$

Then

$$(u, v) = R_\theta(x, y) + (a_1, a_2)$$

and some re-arrangement, together with the fact that $R_\theta^{-1} = R_{-\theta}$, shows that

$$(x, y) = R_{-\theta}(u, v) - R_{-\theta}(a_1, a_2).$$

Hence $T_{\theta,a}^{-1} = T_{-\theta,-R_{-\theta}(a_1,a_2)}$, which is an element of G.

(ii) The rotations about the origin have the form $T_{\theta,0}$. It is easy to check that $T_{\theta,0}T_{\theta',0} = T_{\theta+\theta',0}$ so that the composition of two rotations is another rotation. The identity map is a rotation (through angle 0). The inverse of rotation by θ is rotation by $-\theta$. Hence the set of rotations is a subgroup of G.

(iii) The translations have the form $T_{0,a}$ where $a \in \mathbb{R}^2$. It is easy to see that $T_{0,a}T_{0,a'} = T_{0,a+a'}$ so that the composition of two translations is another translation. The identity map is a translation (by $(0,0)$). The inverse of translation by (a_1, a_2) is translation by $(-a_1, -a_2)$. Hence the set of translations is a subgroup of G.

Solution 2.1

(i) The path determined by both σ_1 and σ_2 is a horizontal line from i to $2+i$.

(ii) We first calculate $\int_\sigma f$ along tha path σ using the parametrisation σ_1. Note that $\sigma_1'(t) = 1$ and $\mathrm{Im}(\sigma_1(t)) = 1$. Hence

$$\begin{aligned}\int_\sigma f &= \int_0^2 f(\sigma_1(t))|\sigma_1'(t)|\,dt \\ &= \int_0^2 dt \\ &= 2.\end{aligned}$$

Now we calculate $\int_\sigma f$ along the path σ using the parametrisation σ_2. Note that $\sigma_2'(t) = 2t-1$ and $\mathrm{Im}(\sigma_2(t)) = 1$. Hence

$$\begin{aligned}\int_\sigma f &= \int_1^2 f(\sigma_2(t))|\sigma_2'(t)|\,dt \\ &= \int_1^2 2t-1\,dt \\ &= t^2 - t\big|_{t=1}^2 \\ &= (4-2)-(1-1) \\ &= 2.\end{aligned}$$

In this example, calculating $\int_\sigma f$ using the second parametrisation was only marginally harder than using the first parametrisation. For more complicated paths, the choice between a 'good' and a 'bad' parametrisation can make the difference between an integral that is easy to calculate and one that is impossible using standard functions!

Solution 2.2

(i) Choose $\sigma : [a,1] \to \mathbb{H}$ given by $\sigma(t) = it$. Then clearly $\sigma(a) = ia$ and $\sigma(1) = i$ (so that $\sigma(\cdot)$ has the required end-points) and $\sigma(t)$ belongs to the imaginary axis. (Note there are many choices of parametrisations, your answer is correct as long as your parametrisation has the correct end-points and belongs to the imaginary axis.)

(ii) For the parametrisation given above, $|\sigma'(t)| = 1$ and $\mathrm{Im}(\sigma(t)) = t$. Hence

$$\mathrm{length}_{\mathbb{H}}(\sigma) = \int_a^1 \frac{1}{t}\,dt = \log t\big|_a^1 = -\log a = \log 1/a.$$

Solution 2.3
The idea is simple: The distance between two points is the infimum of the (hyperbolic) lengths of (piecewise continuously differentiable) paths between them. Only a subset of these paths pass through a third point; hence the infimum of this subset is greater than the infimum over all paths.

Let $x, y, z \in \mathbb{H}$. Let $\sigma_{x,y} : [a, b] \to \mathbb{H}$ be a path from x to y and let $\sigma_{y,z} : [b, c] \to \mathbb{H}$ be a path from y to z. Then the path $\sigma_{x,z} : [a, c] \to \mathbb{H}$ formed by defining

$$\sigma_{x,z}(t) = \begin{cases} \sigma_{x,y}(t) & \text{for } t \in [a, b] \\ \sigma_{y,z}(t) & \text{for } t \in [b, c] \end{cases}$$

is a path from x to z and has length equal to the sum of the lengths of $\sigma_{x,y}, \sigma_{y,z}$. Hence

$$d_{\mathbb{H}}(x, z) \leq \text{length}_{\mathbb{H}}(\sigma_{x,z}) = \text{length}_{\mathbb{H}}(\sigma_{x,y}) + \text{length}_{\mathbb{H}}(\sigma_{y,z}).$$

Taking the infima over path from x to y and from y to z we see that $d_{\mathbb{H}}(x, z) \leq d_{\mathbb{H}}(x, y) + d_{\mathbb{H}}(y, z)$.

Solution 3.1
For a straight line we have $\alpha = 0$, i.e. $\beta z + \bar{\beta}\bar{z} + \gamma = 0$.

Recall that the line $ax + by + c = 0$ has gradient $-a/b$, x-intercept $-c/a$ and y-intercept $-c/b$. Let $z = x + iy$ so that $x = (z + \bar{z})/2$ and $y = (z - \bar{z})/2i$. Substituting these into $ax + by + c$ we see that $\beta = (a - ib)/2$ and $\gamma = c$. Hence the gradient is $\text{Re}(\beta)/\text{Im}(\beta)$, the x-intercept is at $-\gamma/2\text{Re}(\beta)$ and the y-intercept is at $\gamma/2\,\text{Im}(\beta)$.

Solution 3.2
A circle with centre z_0 and radius r has equation $|z - z_0|^2 - r^2 = 0$. Multiplying this out (see the proof of Proposition 3.3.1) we have:

$$z\bar{z} - \bar{z}_0 z - z_0\bar{z} + |z_0|^2 - r^2 = 0$$

and multiplying by $\alpha \in \mathbb{R}$ we have

$$\alpha z\bar{z} - \alpha\bar{z}_0 z - \alpha z_0\bar{z} + \alpha|z_0|^2 - \alpha r^2 = 0.$$

Comparing the coefficients of this with $\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0$ we see that $\beta = -\alpha\bar{z}_0$ and $\gamma = \alpha|z_0|^2 - \alpha r^2$. Hence the centre of the circle is $z_0 = -\bar{\beta}/\alpha$ and the radius is given by

$$r = \sqrt{|z_0|^2 - \frac{\gamma}{\alpha}} = \sqrt{\frac{|\beta|^2}{\alpha^2} - \frac{\gamma}{\alpha}}.$$

Solution 3.3
We first show that γ maps $\mathbb{H}$ to itself, i.e. if $z \in \mathbb{H}$ then $\gamma(z) \in \mathbb{H}$. To see this, let $z = u + iv \in \mathbb{H}$. Then $\text{Im}(z) = v > 0$. Let $\gamma(z) = (az + b)/(cz + d)$ be a Möbius transformation of $\mathbb{H}$. Then

$$\gamma(z) = \frac{a(u + iv) + b}{c(u + iv) + d} = \frac{(au + b + iav)}{(cu + d + icv)}\frac{(cu + d - icv)}{(cu + d - icv)},$$

which has imaginary part

$$\frac{1}{|cz + d|^2}(-cv(au + b) + (cu + d)av) = \frac{1}{|cz + d|^2}(ad - bc)v$$

which is positive. Hence γ maps $\mathbb{H}$ to itself.

If $\gamma(z) = (az+b)/(cz+d)$ then letting $w = (az+b)/(cz+d)$ and solving for z in terms of w shows that $\gamma^{-1}(z) = (dz-b)/(-cz+a)$. Hence γ^{-1} exists and so γ is a bijection.

Solution 3.4

(i) If $\gamma_1 = (a_1z+b_1)/(c_1z+d_1)$ and $\gamma_2 = (a_2z+b_2)/(c_2z+d_2)$ then their composition is

$$\begin{aligned}\gamma_2\gamma_1(z) &= \frac{a_2\left(\frac{a_1z+b_1}{c_1z+d_1}\right)+b_2}{c_2\left(\frac{a_1z+b_1}{c_1z+d_1}\right)+d_2} \\ &= \frac{(a_2a_1+b_2c_1)z+(a_2b_1+b_2d_1)}{(c_2a_1+d_2c_1)z+(c_2b_1+d_2d_1)},\end{aligned}$$

which is a Möbius transformation of $\mathbb{H}$ as

$$\begin{aligned}&(a_2a_1+b_2c_1)(c_2b_1+d_2d_1)-(a_2b_1+b_2d_1)(c_2a_1+d_2c_1) \\ &= (a_1d_1-b_1c_1)(a_2d_2-b_2c_2) > 0.\end{aligned}$$

(ii) Composition of functions is associative.

(iii) The identity map $z \mapsto z$ is a Möbius transformation of $\mathbb{H}$ (take $a=d=1, b=c=0$).

(iv) It follows from the solution to Exercise 3.3 that if γ is a Möbius transformation of $\mathbb{H}$ then so is γ^{-1}.

Solution 3.5

Let $\gamma(z) = (az+b)/(cz+d)$.

For the dilation $z \mapsto kz$ take $a=k, b=0, c=0, d=1$. Then $ad-bc=k>0$ so that γ is a Möbius transformation of $\mathbb{H}$.

For the translation $z \mapsto z+b$ take $a=0, b=b, c=0, d=1$. Then $ad-bc=1>0$ so that γ is a Möbius transformation of $\mathbb{H}$.

For the inversion $z \mapsto -1/z$ take $a=0, b=-1, c=1, d=0$. Then $ad-bc=1>0$ so that γ is a Möbius transformation of $\mathbb{H}$.

Exercise 3.6

Let A be either an arc of circle in $\mathbb{H}$ or part of a straight line in $\mathbb{H}$. Let $\gamma \in \text{Möb}(\mathbb{H})$. Show that $\gamma(A)$ is also either an arc of circle in $\mathbb{H}$ or part of a straight line in $\mathbb{H}$.

Solution 3.6

Let $\gamma \in \text{Möb}(\mathbb{H})$. We know that A is contained in either a circle or straight line in $\mathbb{C}$, and so can be described as

$$A = \{z \in \mathbb{H} \mid \alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0\}$$

for some $\alpha, \gamma \in \mathbb{R}$ and $\beta \in \mathbb{C}$. We need to show that $\gamma(A) = \{z \in \mathbb{H} \mid \alpha' z\bar{z} + \beta' z + \bar{\beta}'\bar{z} + \gamma' = 0\}$ for (possibly different) $\alpha', \gamma' \in \mathbb{R}$ and $\beta' \in \mathbb{C}$.

We know that γ maps $\mathbb{H}$ to $\mathbb{H}$. Hence it is sufficient to prove that if z solves $\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0$ then $\gamma(z)$ solves $\alpha' z\bar{z} + \beta' z + \bar{\beta}'\bar{z} + \gamma' = 0$.

Write $\gamma(z) = (az+b)/(cz+d)$ where $a,b,c,d \in \mathbb{R}$ and $ad-bc > 0$. Let $w = \gamma(z)$. Then $z = \gamma^{-1}(w) = (dw-b)/(-cw+a)$.

Suppose that z solves $\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0$. Then w solves

$$\alpha\left(\frac{dw-b}{-cw+a}\right)\left(\frac{d\bar{w}-b}{-c\bar{w}+a}\right) + \beta\left(\frac{dw-b}{-cw+a}\right) + \bar{\beta}\left(\frac{d\bar{w}-b}{-c\bar{w}+a}\right) + \gamma = 0.$$

Hence

$$\begin{aligned} &\alpha(dw-b)(d\bar{w}-b) + \beta(dw-b)(-c\bar{w}+a) \\ &\quad +\bar{\beta}(d\bar{w}-b)(-cw+a) + \gamma(-cw+a)(-c\bar{w}+a) = 0. \end{aligned}$$

Expanding this out and gathering together terms gives

$$\begin{aligned} &(\alpha d^2 - (\beta+\bar{\beta})cd + \gamma c^2)w\bar{w} + (-\alpha bd + \beta ad + \bar{\beta}bc - \gamma ac)w \\ &\quad +(-\alpha bd + \bar{\beta}ad + \beta bc - \gamma ac)\bar{w} + (\alpha b^2 - (\beta+\bar{\beta})ab + \gamma a^2) = 0. \end{aligned} \tag{25.1}$$

Let

$$\begin{aligned} \alpha' &= \alpha d^2 - (\beta+\bar{\beta})cd + \gamma c^2 \\ \beta' &= -\alpha bd + \beta ad + \bar{\beta}bc - \gamma ac \\ \gamma' &= \alpha b^2 - (\beta+\bar{\beta})ab + \gamma a^2. \end{aligned}$$

Recall that $\beta + \bar{\beta} = 2\,\mathrm{Re}(\beta)$, a real number. Hence α', γ' are real. Hence w satisfies an equation of the form $\alpha' w\bar{w} + \beta' w + \beta'\bar{w} + \gamma'$ with $\alpha', \beta', \gamma' \in \mathbb{R}$, which is the equation of either a vertical line or a circle with real centre.

Solution 4.1

To see that γ maps $\partial\mathbb{H}$ to itself bijectively, it is sufficient to find an inverse. Notice that $\gamma^{-1}(z) = (dz-b)/(-cz+a)$ (defined appropriately for $z = \infty$, namely we set $\gamma^{-1}(\infty) = -d/c$) is an inverse for γ.

Solution 4.2

Let $\gamma(z) = (az+b)/(cz+d)$. Then

$$\gamma'(z) = \frac{(cz+d)a - (az+b)c}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2}$$

so that

$$|\gamma'(z)| = \frac{ad-bc}{|cz+d|^2}.$$

To calculate the imaginary part of $\gamma(z)$, write $z = x+iy$. Then

$$\gamma(z) = \frac{a(x+iy)+b}{c(x+iy)+d} = \frac{(ax+b+iay)}{(cx+d+icy)}\frac{(cx+d-icy)}{(cx+d-icy)},$$

which has imaginary part

$$\begin{aligned} \mathrm{Im}\,\gamma(z) &= \frac{1}{|cz+d|^2}(-cy(ax+b) + (cx+d)ay) \\ &= \frac{1}{|cz+d|^2}(ad-bc)y \\ &= \frac{1}{|cz+d|^2}(ad-bc)\,\mathrm{Im}(z). \end{aligned}$$

Solution 4.3

Let $z = x+iy$ and define $\gamma(z) = -x+iy$.

(i) Suppose that $\gamma(z_1) = \gamma(z_2)$. Write $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Then $-x_1 + iy_1 = -x_2 + iy_2$. Hence $x_1 = x_2$ and $y_1 = y_2$, so that $z_1 = z_2$. Hence γ is injective. Let $z = x + iy \in \mathbb{H}$. take $w = -x + iy$. Then $\gamma(w) = -(-x) + iy = x + iy = z$. Hence γ is surjective. Hence γ is a bijection.

(ii) Let $\sigma(t) = \sigma_1(t) + i\sigma_2(t) : [a, b] \to \mathbb{H}$ be a piecewise continuously differentiable path in $\mathbb{H}$. Note that

$$\gamma \circ \sigma(t) = -\sigma_1(t) + i\sigma_2(t).$$

Hence

$$\begin{aligned} \text{length}_{\mathbb{H}}(\gamma \circ \sigma) &= \int_a^b \frac{1}{\operatorname{Im} \gamma \circ \sigma(t)} \sqrt{(-\sigma_1'(t))^2 + (\sigma_2'(t))^2}\, dt \\ &= \int_a^b \frac{1}{\sigma_2(t)} |\sigma'(t)|\, dt \\ &= \text{length}_{\mathbb{H}}(\sigma). \end{aligned}$$

Let $z, w \in \mathbb{H}$. Note that σ is a piecewise continuously differentiable path from z to w if and only if $\gamma \circ \sigma$ is a piecewise continuously differentiable path from $\gamma(z)$ to $\gamma(w)$. Hence

$$\begin{aligned} d_{\mathbb{H}}(\gamma(z), \gamma(w)) &= \inf\{\text{length}_{\mathbb{H}}(\gamma \circ \sigma) \mid \sigma \text{ is a piecewise continuously} \\ &\qquad \text{differentiable path from } z \text{ to } w\} \\ &= \inf\{\text{length}_{\mathbb{H}}(\sigma) \mid \sigma \text{ is a piecewise continuously} \\ &\qquad \text{differentiable path from } z \text{ to } w\} \\ &= d_{\mathbb{H}}(z, w). \end{aligned}$$

Hence γ is an isometry of $\mathbb{H}$.

Solution 4.4

Let $H_1, H_2 \in \mathcal{H}$. Then there exists $\gamma_1 \in \text{Möb}(\mathbb{H})$ such that $\gamma_1(H_1)$ is the imaginary axis. Similarly, there exists $\gamma_2 \in \text{Möb}(\mathbb{H})$ such that $\gamma_2(H_2)$ is the imaginary axis. Hence γ_2^{-1} maps the imaginary axis to H_2. Hence $\gamma_2^{-1} \circ \gamma_1$ is a Möbius transformation of $\mathbb{H}$ that maps H_1 to H_2.

Solution 5.1

By Lemma 5.2.1 we can find a Möbius transformation γ_1 of $\mathbb{H}$ that maps H_1 to the imaginary axis and z_1 to i and a Möbius transformation γ_2 of $\mathbb{H}$ that maps H_2 to the imaginary axis and z_2 to i. The composition of two Möbius transformations of $\mathbb{H}$ is a Möbius transformation of $\mathbb{H}$. Hence $\gamma_2^{-1} \circ \gamma_1$ is a Möbius transformation of $\mathbb{H}$ that maps H_1 to H_2 and z_1 to z_2.

Solution 5.2

(i) The geodesic between $-3 + 4i$ to $-3 + 5i$ is the arc of vertical straight line between them. It has equation $z + \bar{z} + 6 = 0$.

(ii) Both $-3 + 4i$ and $3 + 4i$ lie on the circle in $\mathbb{C}$ with centre 0 and radius 5. Hence the geodesic between $-3 + 4i$ and $3 + 4i$ is the arc of semi-circle of radius 5 centre 0 between them. It has equation $z\bar{z} - 5^2 = 0$.

(iii) Clearly the geodesic between $-3+4i$ and $5+12i$ is not a vertical straight line. Hence it must have an equation of the form $z\bar{z}+\beta z+\beta\bar{z}+\gamma=0$. Substituting the two values of $z=-3+4i, 5+12i$ we obtain two simultaneous equations:

$$25-6\beta+\gamma=0, 169+10\beta+\gamma=0$$

which can be solved to give $\beta=-9, \gamma=-79$.

Solution 5.3

(i) Let γ be a Möbius transformation of $\mathbb{H}$. As γ is an isometry, by Proposition 4.1.2 we know that

$$\cosh d_{\mathbb{H}}(\gamma(z),\gamma(w)) = \cosh d_{\mathbb{H}}(z,w).$$

Hence $\mathrm{LHS}(\gamma(z),\gamma(w)) = \mathrm{LHS}(z,w)$.

By Exercise 4.2 we know that if γ is a Möbius transformation then $\mathrm{Im}(\gamma(z)) = |\gamma'(z)|\,\mathrm{Im}(z)$. By Lemma 5.5.1 it follows that

$$\begin{aligned} 1+\frac{|\gamma(z)-\gamma(w)|^2}{2\,\mathrm{Im}(\gamma(z))\,\mathrm{Im}(\gamma(w))} &= 1+\frac{|z-w|^2|\gamma'(z)||\gamma'(w)|}{2|\gamma'(z)|\,\mathrm{Im}(z)|\gamma'(w)|\,\mathrm{Im}(w)} \\ &= 1+\frac{|z-w|^2}{2\,\mathrm{Im}(z)\,\mathrm{Im}(w)}. \end{aligned}$$

Hence $\mathrm{RHS}(\gamma(z),\gamma(w)) = \mathrm{RHS}(z,w)$.

(ii) Let H be the geodesic passing through z and w. Then by Lemma 4.3.1 there exists a Möbius transformation γ of $\mathbb{H}$ mapping H to the imaginary axis. Let $\gamma(z)=ia$ and $\gamma(w)=ib$. By interchanging z and w if necessary, we can assume that $a<b$. Then

$$\begin{aligned} \mathrm{LHS}(\gamma(z),\gamma(w)) &= \cosh d_{\mathbb{H}}(\gamma(z),\gamma(w)) \\ &= \cosh d_{\mathbb{H}}(ia,ib) \\ &= \cosh\log b/a \\ &= \frac{e^{\log b/a}+e^{\log a/b}}{2} \\ &= \frac{b/a+a/b}{2} = \frac{b^2+a^2}{2ab}. \end{aligned}$$

Moreover,

$$\begin{aligned} \mathrm{RHS}(\gamma(z),\gamma(w)) &= \mathrm{RHS}(ia,ib) \\ &= 1+\frac{|ia-ib|^2}{2ab} \\ &= 1+\frac{(b-a)^2}{2ab} \\ &= \frac{b^2+a^2}{2ab}. \end{aligned}$$

Hence $\mathrm{LHS}(\gamma(z),\gamma(w)) = \mathrm{RHS}(\gamma(z),\gamma(w))$.

(iii) For any two points z, w let H denote the geodesic containing both z, w. Choose a Möbius transformation γ of $\mathbb{H}$ that maps H to the imaginary axis. Then

$$\mathrm{LHS}(z,w) = \mathrm{LHS}(\gamma(z),\gamma(w)) = \mathrm{RHS}(\gamma(z),\gamma(w)) = \mathrm{RHS}(z,w).$$

Solution 5.4

Let $C = \{w \in \mathbb{H} \mid d_{\mathbb{H}}(z,w) = r\}$ be a hyperbolic circle with centre $z \in \mathbb{H}$ and radius $r > 0$. Recall

$$\cosh d_{\mathbb{H}}(z,w) = 1 + \frac{|z-w|^2}{2\,\mathrm{Im}(z)\,\mathrm{Im}(w)}.$$

Let $z = x_0 + iy_0$ and $w = x + iy$. Then

$$\cosh r = 1 + \frac{(x-x_0)^2 + (y-y_0)^2}{2y_0 y}$$

which can be simplified to

$$(x-x_0)^2 + (y - y_0\cosh r)^2 + y_0^2 - y_0^2\cosh^2 r = 0$$

which is the equation of a Euclidean circle with centre $(x_0, y_0\cosh r)$ and radius $y_0\sqrt{\cosh^2 r - 1} = y_0 \sinh r$.

Solution 5.5

(i) Let $\sigma : [a,b] \to \mathbb{H}$ be any piecewise continuously differentiable path. As we are assuming $\mathrm{length}_\rho(\sigma) = \mathrm{length}_\rho(\gamma\circ\sigma)$ we have

$$\begin{aligned}
\int_a^b \rho(\sigma(t))|\sigma'(t)|\,dt &= \mathrm{length}_\rho(\sigma) \\
&= \mathrm{length}_\rho(\gamma\circ\sigma) \\
&= \int_a^b \rho(\gamma(\sigma(t)))|(\gamma(\sigma(t)))'|\,dt \\
&= \int_a^b \rho(\gamma(\sigma(t)))|\gamma'(\sigma(t))||\sigma'(t)|\,dt
\end{aligned}$$

where we have used the chain rule to obtain the last equality. Hence

$$\int_a^b \left(\rho(\gamma(\sigma(t)))|\gamma'(\sigma(t))| - \rho(\sigma(t))\right)|\sigma'(t)|\,dt = 0.$$

Using the hint, we see that

$$\rho(\gamma(z))|\gamma'(z)| = \rho(z) \tag{25.2}$$

for all $z \in \mathbb{H}$.

(ii) Take $\gamma(z) = z + b$ in (25.2). Then $|\gamma'(z)| = 1$. Hence

$$\rho(z+b) = \rho(z)$$

for all $b \in \mathbb{R}$. Hence $\rho(z)$ depends only the imaginary part of z. Write $\rho(z) = \rho(y)$ where $z = x + iy$.

(iii) Take $\gamma(z) = kz$ in (25.2). Then $|\gamma'(z)| = k$. Hence

$$k\rho(ky) = \rho(y).$$

Setting $y = 1$ and letting $c = \rho(1)$ we have that $\rho(k) = \rho(1)/k = c/k$. Hence $\rho(z) = c/\operatorname{Im}(z)$.

Solution 5.6

(i) Draw in the tangent lines to the circles at the point of intersection; then θ is the angle between these two tangent lines.

Draw the (Euclidean!) triangle with vertices at the point of intersection and the two centres. See Figure 25.1. The internal angle of this triangle at the point of intersection is split into three; the middle part is equal to θ. Recall that a radius of a circle meets the tangent to a circle at right-angles. Hence both the remaining two parts of the angle in the triangle at the point of intersection is given by $\pi/2-\theta$. Hence the triangle has angle $\pi/2 - \theta + \theta + \pi/2 - \theta = \pi - \theta$ at the vertex corresponding to the point of intersection.

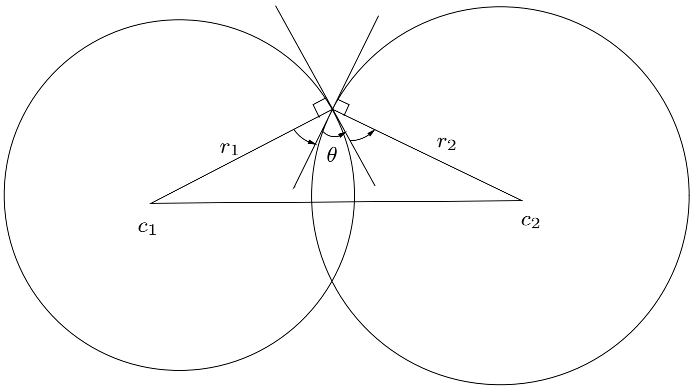

Figure 25.1: The Euclidean triangle with vertices at c_1, c_2 and the point of intersection.

The cosine rule gives the required formula (recall that $\cos \pi - \theta = -\cos\theta$).

(ii) The points -6 and 6 clearly lie on the semi-circle with centre 0 and radius 6. Similarly, the points $4\sqrt{2}$ and $6\sqrt{2}$ clearly lies on the semi-circle with centre $5\sqrt{2}$ and radius $\sqrt{2}$.

(If you can't determine the geodesic by considering the geometry then you can find it as follows. We know geodesics have equations of the form $\alpha z\bar{z} + \beta z + \beta\bar{z} + \gamma = 0$ where $\alpha, \beta, \gamma \in \mathbb{R}$. The geodesic between $4\sqrt{2}$ and $6\sqrt{2}$ is clearly a semi-circle, and so $\alpha \neq 0$; we divide through by α to assume that $\alpha = 1$. Hence we are looking for an equation of the form $z\bar{z} + \beta z + \beta\bar{z} + \gamma = 0$. Substituting first $z = 4\sqrt{2}$ and then $z = 6\sqrt{2}$ we obtain the simultaneous equations $32+8\sqrt{2}\beta+\gamma = 0$, $72+12\sqrt{2}\beta+\gamma = 0$. Solving these gives $\beta = -5\sqrt{2}$, $\gamma = 48$. Putting $z = x + iy$ we thus have the equation $x^2 + y^2 - 10\sqrt{2}x + 48 = 0$. Completing the square gives $(x - 5\sqrt{2})^2 + y^2 = (\sqrt{2})^2$, so that we have a semi-circle in $\mathbb{C}$ with centre $5\sqrt{2}$ and radius $\sqrt{2}$.)

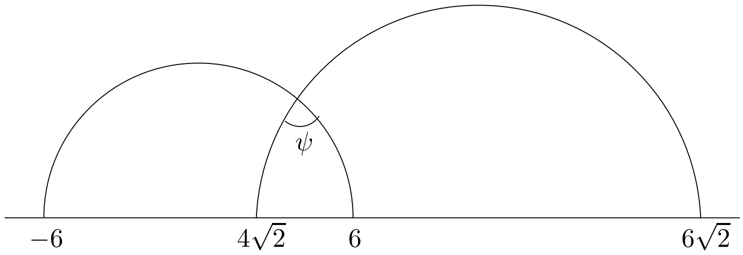

Figure 25.2: The angle ψ.

Part (i) allows us to calculate the angle ψ in Figure 25.2. Substituting $c_1 = 0$, $r_1 = 6$, $c_2 = 5\sqrt{2}$, $r_2 = \sqrt{2}$ into the result from (i) shows that $\cos\psi = 1/\sqrt{2}$ so that $\psi = \pi/4$. The angle in Figure 5.6 that we want to calculate is $\phi = \pi - \psi = 3\pi/4$.

Solution 5.7

Suppose that the semi-circular geodesic has centre at $x \in \mathbb{R}$ and radius r. Construct the (Euclidean) right-angled triangle with vertices $x, 0, ib$, as illustrated in Figure 25.3. As

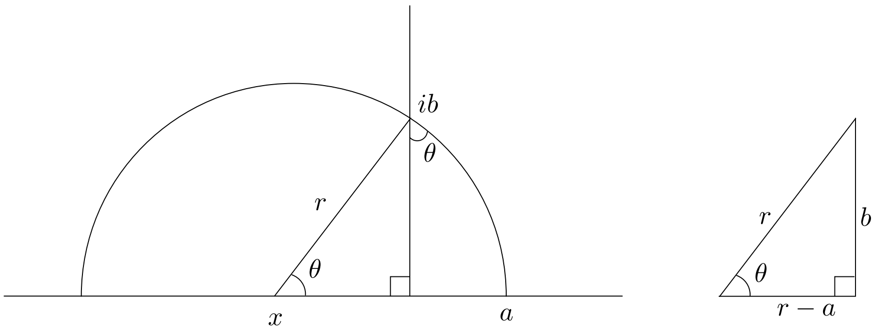

Figure 25.3: The (Euclidean) triangle with vertices at $x, 0, ib$.

the radius of the semicircle is r, we have that $|x - ib| = r$ and $|x - a| = r$; hence the base of the right-angled triangle has length $r - a$. By Pythagoras' Theorem, we have that $(r-a)^2 + b^2 = r^2$. Expanding this out and simplyfying it we have $r = (a^2 + b^2)/2a$. From Figure 25.3 we also have that

$$\sin\theta = \frac{b}{r}, \quad \cos\theta = \frac{r-a}{r}$$

and the result follows after substituting in $r = (a^2 + b^2)/2a$.

Solution 6.1

(i) First note that h is a bijection from $\mathbb{H}$ to its image because it has an inverse $g(z) = (-z + i)/(-iz + 1)$.

We now show that $h(\mathbb{H}) = \mathbb{D}$. Let $z = u + iv \in \mathbb{H}$ so that $v > 0$. Now

$$h(z) \quad = \quad \frac{u + iv - i}{i(u + iv) - 1}$$

$$\begin{aligned} &= \frac{u+i(v-1)}{-(v+1)+iu}\frac{-(v+1)-iu}{-(v+1)-iu} \\ &= \frac{-2u+i(1-u^2-v^2)}{(v+1)^2+u^2}. \end{aligned}$$

To show that $h(\mathbb{H}) = \mathbb{D}$ it remains to show that the above complex number has modulus less than 1. To see this first note that:

$$\begin{aligned} &(2u)^2 + (1-u^2-v^2)^2 \\ &\quad = u^4 + 2u^2 + 1 - 2v^2 + 2u^2v^2 + v^4 \qquad (25.3) \\ &((v+1)^2+u^2)^2 \\ &\quad = v^4 + 4v^3 + 6v^2 + 4v + 1 \\ &\qquad + 2u^2v^2 + 4u^2v + 2u^2 + u^4. \qquad (25.4) \end{aligned}$$

To prove that $|h(z)| < 1$ it is sufficient to check that (25.3) < (25.4). By cancelling terms, it is sufficient to check that

$$-2v^2 < 4v^3 + 6v^2 + 4v + 4u^2v.$$

This is true because the left-hand side is clearly negative, whereas the right-hand side is positive, using the fact that $v > 0$.

To show that h maps $\partial\mathbb{H}$ bijectively to $\partial\mathbb{D}$ note that for $u \in \mathbb{R}$

$$h(u) = \frac{-2u + i(1-u^2)}{u^2+1}$$

which is easily seen to have modulus one (and so is a point on $\partial\mathbb{D}$). Note that $h(\infty) = -i$ and that $h(u) \neq -i$ if u is real. Hence h is a bijection from $\partial\mathbb{H}$ to $\partial\mathbb{D}$.

(ii) We have already seen that $g(z) = h^{-1}(z) = (-z+i)/(-iz+1)$. Calculating $g'(z)$ is easy. To calculate $\mathrm{Im}(g(z))$ write $z = u+iv$ and compute.

(iii) Let $\sigma(t) = t$, $0 \leq t \leq x$. Then σ is a path from 0 to x and it has length

$$\begin{aligned} \int_\sigma \frac{2}{1-|z|^2} &= \int_0^x \frac{2}{1-t^2}\,dt \\ &= \int_0^x \frac{1}{1-t} + \frac{1}{1+t}\,dt \\ &= \log\frac{1+x}{1-x}. \end{aligned}$$

To show that this is the optimal length of a path from 0 to x (and thus that the real-axis is a geodesic) we have to show that any other path from 0 to x has a larger length.

Let $\sigma(t) = x(t) + iy(t)$, $a \leq t \leq b$ be a path from 0 to x. Then it has length

$$\begin{aligned} &\int_a^b \frac{2}{1-|\sigma(t)|^2}|\sigma'(t)|\,dt \\ &\quad = \int_a^b \frac{2}{1-(x(t)^2+y(t)^2)}\sqrt{x'(t)^2+y'(t)^2}\,dt \end{aligned}$$

$$\begin{aligned}
&\geq \int_a^b \frac{2}{1-x(t)^2}x'(t)\,dt \\
&= \int_a^b \frac{x'(t)}{1-x(t)} + \frac{x'(t)}{1+x(t)}\,dt \\
&= \log \frac{1+x(t)}{1-x(t)}\bigg|_a^b \\
&= \log \frac{1+x}{1-x},
\end{aligned}$$

with equality precisely when $y'(t) = 0$ and $y(t) = 0$, i.e. with equality precisely when the path lies along the real axis.

Solution 6.2
Recall $h(z) = (z-i)/(iz-1)$ and $h^{-1}(z) = (-z+i)/(-iz+1)$. Let $\gamma(z) = (az+b)/(cz+d)$, $ad-bc > 0$, be a Möbius transformation of $\mathbb{H}$. We claim that $h\gamma h^{-1}$ is a Möbius transformation of $\mathbb{D}$.

To see this, first note that (after a *lot* of algebra!)

$$\begin{aligned}
h\gamma h^{-1}(z) &= \frac{[a+d+i(b-c)]z + [-(b+c)-i(a-d)]}{[-(b+c)+i(a-d)]z + [a+d-i(b-c)]} \\
&= \frac{\alpha z+\beta}{\bar{\beta}z+\bar{\alpha}}.
\end{aligned}$$

Finally, we must check that $|\alpha|^2 - |\beta|^2 > 0$ which is a simple calculation, using the fact that $ad-bc > 0$.

Solution 6.3
Let

$$\gamma_1(z) = \frac{\alpha_1 z+\beta_1}{\bar{\beta}_1 z+\bar{\alpha}_1}, \quad \gamma_2(z) = \frac{\alpha_2 z+\beta_2}{\bar{\beta}_2 z+\bar{\alpha}_2}$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{C}$ and $|\alpha_1|^2 - |\beta_1|^2 = 1$ and $|\alpha_2|^2 - |\beta_2|^2 = 1$. We want to show that $\gamma_1\gamma_2 \in \text{Möb}(\mathbb{D})$. Note that

$$\begin{aligned}
\gamma_1(\gamma_2(z)) &= \frac{\alpha_1\gamma_2(z)+\beta_1}{\bar{\beta}_1\gamma_2(z)+\bar{\alpha}_1} \\
&= \frac{\alpha_1\left(\frac{\alpha_2 z+\beta_2}{\bar{\beta}_2 z+\bar{\alpha}_2}\right)+\beta_1}{\bar{\beta}_1\left(\frac{\alpha_2 z+\beta_2}{\bar{\beta}_2 z+\bar{\alpha}_2}\right)+\bar{\alpha}_1} \\
&= \frac{(\alpha_1\alpha_2+\beta_1\bar{\beta}_2)z + (\alpha_1\beta_2+\beta_1\bar{\alpha}_2)}{(\bar{\beta}_1\alpha_2+\bar{\alpha}_1\bar{\beta}_2)z + (\bar{\beta}_1\beta_2+\bar{\alpha}_1\bar{\alpha}_2)} \\
&= \frac{\alpha z+\beta}{\bar{\beta}z+\bar{\alpha}}
\end{aligned}$$

where $\alpha = \alpha_1\alpha_2 + \beta_1\bar{\beta}_2$ and $\beta = \alpha_1\beta_2 + \beta_1\bar{\alpha}_2$. It is straightforward to check that

$$|\alpha|^2 - |\beta|^2 = (|\alpha_1|^2 - |\beta_1|^2)(|\alpha_2|^2 - |\beta_2|^2) > 0.$$

Hence $\gamma_1\gamma_2 \in \text{Möb}(\mathbb{D})$.

Suppose that
$$\frac{\alpha z+\beta}{\bar\beta z+\bar\alpha}=w.$$
Then
$$w=\frac{\bar\alpha z-\beta}{-\bar\beta z+\alpha}.$$
Hence
$$\gamma^{-1}(z)=\frac{\bar\alpha z-\beta}{-\bar\beta z+\alpha}$$
and it is easy to see that $\gamma^{-1}\in \text{Möb}(\mathbb{D})$.

Clearly the identity map $z\mapsto z$ is a Möbius transformation of $\mathbb{D}$ (take $\alpha=1,\beta=0$).

Hence $\text{Möb}(\mathbb{D})$ is a group.

Solution 6.4

The map $h:\mathbb{H}\to\mathbb{D}$ defined in (6.1.1) maps geodesics in $\mathbb{H}$ to geodesics in $\mathbb{D}$.

Suppose that $z\in\mathbb{H}$ lies on a geodesic. Then z lies on either a horizontal straight line or semi-circle with real centre with an equation of the form
$$\alpha z\bar z+\beta z+\beta\bar z+\gamma=0.$$

Let $w=h(z)$. Then
$$z=\frac{-w+i}{-iw+1}$$
so that
$$\bar z=\frac{-\bar w-i}{i\bar w+1}.$$
Hence w satisfies an equation of the form
$$\alpha\frac{-w+i}{-iw+1}\frac{-\bar w-i}{i\bar w+1}+\beta\frac{-w+i}{-iw+1}+\beta\frac{-\bar w-i}{i\bar w+1}+\gamma=0.$$
Equivalently, w satisfies an equation of the form
$$\alpha(-w+i)(-\bar w-i)+\beta(-w+i)(i\bar w+1)+\beta(-\bar w-i)(-iw+1)+\gamma(-iw+1)(i\bar w+1)=0.$$
Multiplying this out and collecting terms we see that w satisfies an equation of the form
$$(\alpha+\gamma)w\bar w+(-2\beta+i(\alpha-\gamma))w+(-2\beta-i(\alpha-\gamma))w+(\alpha+\gamma)=0.$$
Let $\alpha'=\alpha+\gamma$, $\beta'=-2\beta+i(\alpha-\gamma)$. Then $\alpha'\in\mathbb{R}$ and $\beta'\in\mathbb{C}$. Moreover w satisfies an equation of the form
$$\alpha' w\bar w+\beta' w+\bar\beta'\bar w+\alpha'=0.$$

Solution 6.5

By applying a Möbius transformation of $\mathbb{D}$, we can move the circle so that its centre is at the origin $0\in\mathbb{D}$. (This uses the additional facts that (i) a hyperbolic circle is a Euclidean circle (but possibly with a different centre and radius), and (ii) Möbius transformations of $\mathbb{D}$ map circles to circles.) As Möbius transformations of $\mathbb{D}$ preserve lengths and area, this doesn't change the circumference nor the area.

Let $C_r = \{w \in \mathbb{D} \mid d_{\mathbb{D}}(0,w) = r\}$. By Proposition 6.2.1 and the fact that a rotation is a Möbius transformation of $\mathbb{D}$, we have that C_r is a Euclidean circle with centre 0 and radius R where

$$\frac{1+R}{1-R} = e^r.$$

Hence $R = (e^r - 1)/(e^r + 1) = \tanh(r/2)$.

Now

$$\text{circumference}(C_r) = \int_\sigma \frac{2}{1-|z|^2}$$

where $\sigma(t) = Re^{it}$, $0 \le t \le 2\pi$ is a path that describes the Euclidean circle of radius R, centred at 0. Now

$$\begin{aligned}
\text{circumference}(C_r) &= \int_\sigma \frac{2}{1-|z|^2} \\
&= \int_0^{2\pi} \frac{2}{1-|\sigma(t)|^2}|\sigma'(t)|\,dt \\
&= \int_0^{2\pi} \frac{2R}{1-R^2}\,dt \\
&= \frac{4\pi R}{1-R^2}
\end{aligned}$$

and substituting for R in terms of r gives that the circumference of C_r is $2\pi \sinh r$.

Similarly, the area of C_r is given by

$$\text{Area}_{\mathbb{D}}(C_r) = \int\int_{D_r} \frac{4}{(1-|z|^2)^2}\,dz$$

where $D_r = \{w \in \mathbb{D} \mid d_{\mathbb{D}}(0,w) \le r\}$ is the disc of hyperbolic radius r with centre 0. Now D_r is the Euclidean disc of radius $R = \tanh(r/2)$ centred at 0. Recall that when integrating using polar co-ordinates, the area element is $\rho\,d\rho\,d\theta$. Then

$$\begin{aligned}
\text{Area}_{\mathbb{D}}(C_r) &= \int_{\theta=0}^{2\pi}\int_{\rho=0}^{R} \frac{4}{(1-\rho^2)^2}\rho\,d\rho\,d\theta \\
&= 4\pi \left.\frac{1}{1-\rho^2}\right|_{\rho=0}^{R} \\
&= 4\pi\frac{R^2}{1-R^2} \\
&= 4\pi \sinh^2 r/2.
\end{aligned}$$

Solution 7.1

(i) Clearly both $(-1+i\sqrt{3})/2$ and $(1+i\sqrt{3})/2$ lie on the unit circle in $\mathbb{C}$ with centre 0 and radius 1.

One can determine the other two geodesics by recognition; alternatively one can argue as follows. Consider 0, $(-1+i\sqrt{3})/2$. These two points lie on a geodesic given by a semi-circle with equation $z\bar{z} + \beta z + \beta\bar{z} + \gamma = 0$. Substituting these two values of z in gives the simultaneous equations $\gamma = 0$, $1 - \beta + \gamma = 0$. Hence $\beta = 1$, $\gamma = 0$. Hence the equation of the geodesic through 0, $(-1+i\sqrt{3})/2$ is given by $z\bar{z}+z+\bar{z}=0$. Writing $z=x+iy$ this becomes $x^2+y^2+2x=0$. Completing the square gives $(x+1)^2+y^2=1$. Hence 0, $(-1+i\sqrt{3})/2$ lie on the circle in $\mathbb{C}$ with centre -1 and radius 1.

A similar calculation shows that 0, $(1+i\sqrt{3})/2$ lie on the circle in $\mathbb{C}$ with centre 1 and radius 1.

(ii) As the vertex 0 is on the boundary of $\mathbb{H}$, the internal angle is 0.

We can calculate the angles ψ_1, ψ_2 in Figure 25.4 using Exercise 5.6. We obtain

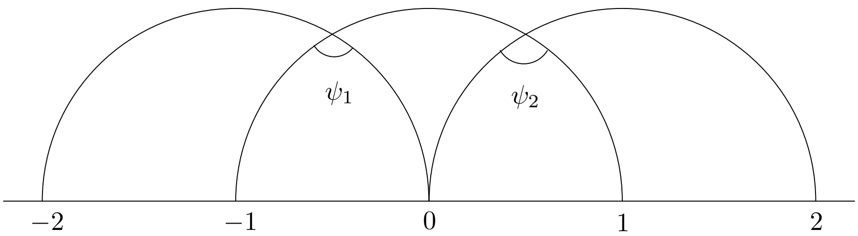

Figure 25.4: The angles ψ_1, ψ_2.

$$\cos\psi_1 = \frac{(0-1)^2-(1^2+1^2)}{2} = -\frac{1}{2}$$

so that $\psi_1 = 2\pi/3$. As $\theta_1 = \pi - \psi_1$ we have $\theta_1 = \pi/3$.

Similarly, $\theta_2 = \pi/3$.

By the Gauss-Bonnet Theorem, the area of the triangle is $\pi-(0+\pi/3+\pi/3)=\pi/3$.

Solution 7.2

Let Q be a hyperbolic quadrilateral with vertices A, B, C, D (labelled, say, anti-clockwise) and corresponding internal angles $\alpha, \beta, \gamma, \delta$. Construct the geodesic from A to C, creating triangles ABC (with internal angles $\alpha_1, \beta, \gamma_1$) and CDA (with internal angles $\gamma_2, \delta, \alpha_2$), where $\alpha_1+\alpha_2=\alpha$ and $\gamma_1+\gamma_2=\gamma$. By the Gauss-Bonnet Theorem

$$\begin{aligned}
\mathrm{Area}_{\mathbb{H}}(Q) &= \mathrm{Area}_{\mathbb{H}}(ABC)+\mathrm{Area}_{\mathbb{H}}(CDA) \\
&= \pi-(\alpha_1+\beta+\gamma_1)+\pi-(\alpha_2+\beta+\gamma_2) \\
&= 2\pi-(\alpha+\beta+\gamma+\delta).
\end{aligned}$$

Solution 7.3

Let $D(r)$ be the hyperbolic polygon with vertices at $r, r\omega, \ldots, r\omega^{n-1}$. Let $\alpha_j(r)$ denote the internal angle at vertex $r\omega^j$. For each $0 \leq k \leq n-1$, consider the Möbius transformation of $\mathbb{D}$ given by $\gamma_k(z) = w^k z$; this rotates the polygon so that vertex v_i is mapped to vertex v_{i+k}. Thus $\gamma_k(D(r)) = D(r)$. As Möbius transformations of $\mathbb{D}$ preserve angles, this shows that the internal angle at vertex v_1 is equal to the internal angle at vertex v_{1+k}. By varying k, we see that all internal angles are equal.

By the Gauss-Bonnet Theorem, we see that

$$\mathrm{Area}_{\mathbb{H}} D(r) = (n-2)\pi - n\alpha(r).$$

Notice that $D(r)$ is contained in $C(r)$, the hyperbolic disc with hyperbolic centre 0 and Euclidean radius r. By (the solution to) Exercise 6.5, we see that

$$\begin{aligned}\lim_{r\to 0}\mathrm{Area}_{\mathbb{H}}D(r) &\leq \lim_{r\to 0}\mathrm{Area}_{\mathbb{H}}C(r)\\ &= \lim_{r\to 0}\frac{4\pi r^2}{1-r^2}=0.\end{aligned}$$

Hence

$$\lim_{r\to 0}\alpha(r)=\frac{(n-2)\pi}{n}.$$

As $r\to 1$, each vertex $r\omega^k\to\omega^k\in\partial\mathbb{D}$. The internal angle at a vertex on the boundary is equal to 0. Hence $\lim_{r\to 1}\alpha(r)=0$.

Hence given any $\alpha\in[0,(n-2)\pi/n)$, we can find a value of r for which $\alpha=\alpha(r)$, and hence construct a regular n-gon with internal angle α.

Conversely, suppose that D is a regular hyperbolic polygon with each internal angle $\alpha\geq(n-2)\pi/n$. Then we have that $n\alpha\geq(n-2)\pi$. By the Gauss-Bonnet Theorem,

$$\mathrm{Area}_{\mathbb{H}}D=(n-2)\pi-n\alpha\leq(n-2)\pi-(n-2)\pi=0.$$

As area must be positive, this is a contradiction.

Solution 7.4
(Not examinable—included for interest only!)

Clearly $n\geq 3$ and $k\geq 3$.

The internal angle of a regular (Euclidean) n-gon is $(n-2)\pi/n$. Suppose that k n-gons meet at each vertex. As the polyhedron is convex, the angle sum must be less than 2π. Hence

$$k\frac{(n-2)\pi}{n}<2\pi.$$

Rearranging this and completing the square gives $(k-2)(n-2)<4$. As n, k are integers greater than 3, we must have that either $n=3$ or $k=3$. It is easy to see that the only possibilities are $(n,k)=(3,3)$, $(3,4)$, $(3,5)$, $(4,3)$ and $(5,3)$, as claimed.

Solution 8.1
First note that

$$\begin{aligned}\cos^2\alpha &= \frac{1}{1+\tan^2\alpha}\\ &= \frac{1}{1+\frac{\tanh^2 a}{\sinh^2 b}}\\ &= \frac{\sinh^2 b}{\sinh^2 b+\tanh^2 a}.\end{aligned}$$

Now using the facts that $\cosh c=\cosh a\cosh b$ and $\tanh^2 a=1-1/\cosh^2 a$ we see that

$$\tanh^2 a=1-\frac{\cosh^2 b}{\cosh^2 c}.$$

Substituting this into the above equality gives

$$\begin{aligned}\cos^2\alpha &= \frac{\sinh^2 b}{\sinh^2 b+1-\frac{\cosh^2 b}{\cosh^2 c}}\\ &= \frac{\tanh^2 b}{\tanh^2 c}\end{aligned}$$

(after some manipulation, using the fact that $\cosh^2 - \sinh^2 = 1$).

To see that $\sin\beta = \sinh b / \sinh c$ we multiply the above equation and the equation given in Proposition 8.2.1 together to obtain

$$\begin{aligned} \sin\alpha &= \frac{\tanh b}{\tanh c}\frac{\tanh a}{\sinh} \\ &= \frac{\sinh b}{\cosh b}\frac{\cosh c}{\sinh c}\frac{\sinh a}{\cosh a}\frac{1}{\sinh b} \\ &= \frac{\sinh a}{\sinh c}, \end{aligned}$$

using the fact that $\cosh c = \cosh a \cosh b$.

Solution 8.2

We prove the first identity. By Proposition 8.2.1 we know that

$$\cos\alpha = \frac{\tanh b}{\tanh c},\ \sin\beta = \frac{\sinh b}{\sinh c}.$$

Hence

$$\frac{\cos\alpha}{\sin\beta} = \frac{\tanh b}{\tanh c}\frac{\sinh c}{\sinh b} = \frac{\cosh c}{\cosh b} = \cosh a,$$

using the hyperbolic version of Pythagoras' Theorem.

We prove the second identity. By Proposition 8.2.1 we have that

$$\tan\alpha = \frac{\tanh a}{\sinh b},\ \tan\beta = \frac{\tanh b}{\sinh a}.$$

Hence

$$\cot\alpha\cot\beta = \frac{\sinh a}{\tanh b}\frac{\sinh b}{\tanh a} = \cosh a \cosh b = \cosh c,$$

by the hyperbolic versin of Pythagoras' Theorem.

Take a Euclidean right-angled triangle with sides of length a, b and c, with c being the hypotenuse. Let α be the angle opposite a and β opposite b. Then $\cos\alpha = b/c$ and $\sin\beta = b/c$ so that

$$\cos\alpha \operatorname{cosec}\beta = 1.$$

As in a Euclidean triangle the angles sum to π, we must have that $\beta = \pi/2 - \alpha$. Hence the above identity says that $\sin(\pi/2 - \alpha) = \cos\alpha$.

Similarly, we have that $\tan\alpha = a/b$ and $\tan\beta = b/a$. Hence

$$\cot\alpha\cot\beta = 1.$$

Again, this can be re-written as $\tan(\pi/2 - \alpha) = 1/\tan\alpha$.

Solution 8.3

Note that

$$\cos\alpha = \sqrt{1 - \sin^2\alpha} = \sqrt{1 - \frac{1}{\cosh^2 a}} = \frac{\sinh a}{\cosh a} = \frac{1}{\tanh a}.$$

Hence

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{1}{\cosh a}\frac{\cosh a}{\sinh a} = \frac{1}{\sinh a}.$$

Solution 8.4
Label the vertices A, B and C so that the angle at A is α, etc. By applying a Möbius transformation of $\mathbb{H}$ we may assume that none of the sides of Δ are segments of vertical lines. Construct a geodesic from vertex B to the geodesic segment $[A, C]$ in such a way that these geodesics meet at right-angles. This splits Δ into two right-angled triangles, BDA and BDC. Let the length of the geodesic segment $[B, D]$ be d, and suppose that BDA has internal angles $\beta_1, \pi/2, \alpha$ and side lengths d, b_1, c, as in the figure. Label BDC similarly. See Figure 25.5.

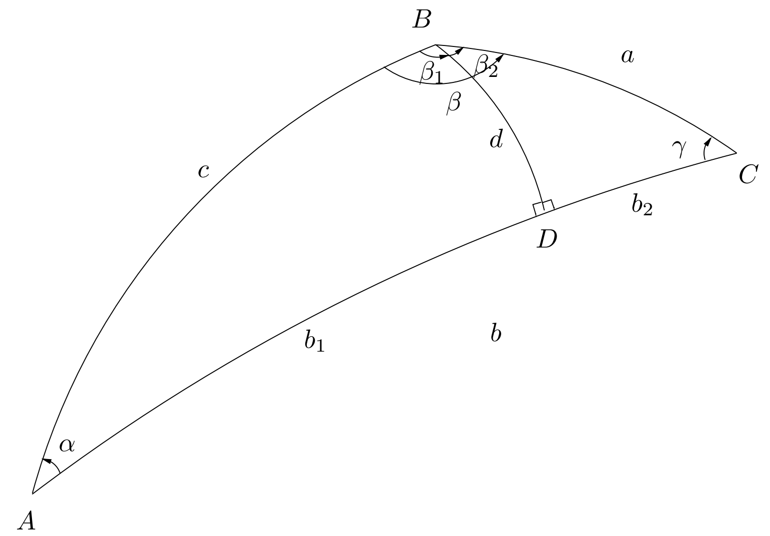

Figure 25.5: The sine rule.

From Proposition 8.2.1 we know that

$$\sin\beta_1 = \frac{\sinh b_1}{\sinh c},\ \cos\beta_1 = \frac{\tanh d}{\tanh c},\ \sin\beta_2 = \frac{\sinh b_2}{\sinh a},\ \cos\beta_2 = \frac{\tanh d}{\tanh a}.$$

By the hyperbolic version of Pythagoras' Theorem we know that

$$\cosh c = \cosh b_1 \cosh d,\ \cosh a = \cosh b_2 \cosh d.$$

Hence

$$\begin{aligned}
\sin\beta &= \sin(\beta_1+\beta_2) \\
&= \sin\beta_1\cos\beta_2 + \sin\beta_2\cos\beta_1 \\
&= \frac{\sinh b_1}{\sinh c}\frac{\sinh d}{\cosh d}\frac{\cosh a}{\sinh a} + \frac{\sinh b_2}{\sinh a}\frac{\sinh d}{\cosh d}\frac{\cosh c}{\sinh c} \\
&= \frac{\sinh b_1 \sinh d}{\sinh c \sinh a}\cosh b_2 + \frac{\sinh b_2 \sinh d}{\sinh a \sinh c}\cosh b_1 \\
&= \frac{\sinh d}{\sinh a \sinh c}(\sinh b_1 \cosh b_2 + \sinh b_2 \cosh b_1) \\
&= \frac{\sinh d}{\sinh a \sinh c}\sinh(b_1+b_2) \\
&= \frac{\sinh b \sinh d}{\sinh a \sinh c}.
\end{aligned}$$

Using Proposition 8.2.1 again, we see that $\sin\alpha = \sinh d/\sinh c$ and $\sin\gamma = \sinh d/\sin a$. Substituting these into the above equality proves the result.

Solution 9.1

γ_1 has one fixed point in $\mathbb{H}$ at $(-3+i\sqrt{51})/6$ and so is elliptic. γ_2 has fixed points at ∞ and -1 and so is hyperbolic. γ_3 has one fixed point at i and so is elliptic. γ_4 has one fixed point at 0 and so is parabolic.

Solution 9.2

We have

$$\gamma_1(z) = \frac{\frac{2}{\sqrt{13}}z + \frac{5}{\sqrt{13}}}{\frac{-3}{\sqrt{13}}z + \frac{-1}{\sqrt{13}}},\ \gamma_2(z) = \frac{\frac{7}{\sqrt{7}}z + \frac{6}{\sqrt{7}}}{\frac{1}{\sqrt{7}}},$$

and γ_3 and γ_4 are already normalised.

Solution 9.3

(i) Clearly the identity is in $\mathrm{SL}(2,\mathbb{R})$. If $A \in \mathrm{SL}(2,\mathbb{R})$ is the matrix $(a,b;c,d)$ then A^{-1} has matrix $(d,-b;-c,a)$, which is in $\mathrm{SL}(2,\mathbb{R})$. If $A,B \in \mathrm{SL}(2,\mathbb{R})$ then $\det AB = \det A \det B = 1$ so that the product $AB \in \mathrm{SL}(2,\mathbb{R})$.

(ii) We show that $\mathrm{SL}(2,\mathbb{Z})$ is a subgroup. Clearly the identity is in $\mathrm{SL}(2,\mathbb{Z})$. If $A,B \in \mathrm{SL}(2,\mathbb{Z})$ then the product matrix AB has entries formed by taking sums and products of the entries of A and B. As the entries of A,B are integers, so are any combination of sums and products of the entries. Hence $AB \in \mathrm{SL}(2,\mathbb{Z})$. Finally, we need to check that if $A \in \mathrm{SL}(2,\mathbb{Z})$ then so is A^{-1}. This is easy, as if $A = (a,b;c,d)$ then $A^{-1} = (d,-b;-c,a)$, which has integer entries.

Solution 10.1

(i) Recall that the Möbius transformation γ_1 of $\mathbb{H}$ is conjugate to the Möbius transformation γ_2 of $\mathbb{H}$ if there exists a Möbius transformation $g \in \text{Möb}(\mathbb{H})$ such that $g\gamma_1 g^{-1} = \gamma_2$.

Clearly γ is conjugate to itself (take $g = \mathrm{id}$).

If $\gamma_2 = g\gamma_1 g^{-1}$ then $\gamma_1 = g^{-1}\gamma_2 g$ so that γ_2 is conjugate to γ_1 if γ_1 is conjugate to γ_2.

If $\gamma_2 = g\gamma_1 g^{-1}$ and $\gamma_3 = h\gamma_2 h^{-1}$ then $\gamma_3 = (hg)\gamma_1(hg)^{-1}$ so that γ_3 is conjugate to γ_1.

(ii) Let γ_1 and γ_2 be conjugate. Write $\gamma_2 = g\gamma_1 g^{-1}$ where $g \in \text{Möb}(\mathbb{H})$. Then

$$\begin{aligned} \gamma_1(x) = x &\Leftrightarrow g^{-1}\gamma_2 g(x) = x \\ &\Leftrightarrow \gamma_2(g(x)) = g(x) \end{aligned}$$

so that x is a fixed point of γ_1 if and only if $g(x)$ is a fixed point of γ_2. Hence g maps the set of fixed points of γ_1 to the set of fixed points of γ_2. As g is a Möbius transformation of $\mathbb{H}$ and therefore a bijection, we see that γ_1 and γ_2 have the same number of fixed points.

Solution 10.2
Let $A = (a_{ij}), B = (b_{ij})$ be two matrices. We first show that trace(AB) = trace(BA). Recall that the trace of a matrix is the sum of the diagonal elements. Hence

$$\begin{aligned} \mathrm{trace}(AB) &= \sum_i (AB)_{ii} \\ &= \sum_i \sum_j a_{ij} b_{ji} = \sum_j \sum_i b_{ji} a_{ij} = \sum_j (BA)_j \\ &= \mathrm{trace}(BA) \end{aligned}$$

where $(AB)_{ij}$ denotes the (i, j)th entry of AB.

Let
$$\gamma_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1}, \ \gamma_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2}$$
be two conjugate Möbius transformations of $\mathbb{H}$. Let

$$A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \ A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix},$$

be their corresponding (normalised) matrices. Let g be a Möbius transformation of $\mathbb{H}$ such that $\gamma_1 = g^{-1}\gamma_2 g$. Suppose that g has matrix A. By replacing A by $-A$ if necessary, it follows from the remarks in Lecture 10 that $A_1 = A^{-1}A_2A$.

Hence
$$\mathrm{trace}(A_1) = \mathrm{trace}(A^{-1}A_2A) = \mathrm{trace}(A_2AA^{-1}) = \mathrm{trace}(A_2).$$
Hence $\tau(\gamma_1) = \mathrm{trace}(A_1)^2 = \mathrm{trace}(A_2)^2 = \tau(\gamma_2)$.

Solution 10.3
Let $\gamma_1(z) = z + b$ where $b > 0$ and let $\gamma_2(z) = z + 1$. As both γ_1 and γ_2 have fixed points at ∞ and a conjugacy acts a 'change of co-ordinates', we look for a conjugacy from γ_1 to γ_2 that fixes ∞. We will try $g(z) = kz$ for some (to be determined) $k > 0$. Now $g^{-1}\gamma_1 g(z) = g^{-1}\gamma_1(kz) = g^{-1}(kz + b) = z + b/k$. So we choose $k = b$.

Now let $\gamma_1(z) = z - b$ where $b > 0$ and let $\gamma_2(z) = z - 1$. Again, let $g(z) = kz$ for some $k > 0$. Then $g^{-1}\gamma_1 g(z) = g^{-1}\gamma(kz) = g^{-1}(kz - b) = z - b/k$. So again we choose $k = b$.

Suppose that $\gamma_1(z) = z + 1$ and $\gamma_2(z) = z - 1$ are conjugate. Then there exists $g(z) = (az + b)/(cz + d) \in$ Möb$(\mathbb{H})$ such that $\gamma_1 g(z) = g\gamma_2(z)$. In terms of matrices, this says that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

That is,
$$\begin{pmatrix} a + c & b + d \\ c & d \end{pmatrix} = \begin{pmatrix} a & -a + b \\ c & -c + d \end{pmatrix}.$$

Comparing coefficient in the '+' case, we see that $c = 0$ and $d = -a$. Hence $ad - bc = -a^2 < 0$, a contradiction. In the '$-$' case, we see that $c = 0$, $d = 0$, so that $ad - bc = 0$, again a contradiction. Hence γ_1, γ_2 are not conjugate in Möb$(\mathbb{H})$.

Solution 11.1
Let $\gamma_1(z) = k_1 z$ and $\gamma_2(z) = k_2 z$ where $k_1, k_2 \neq 1$. Suppose that γ_1 is conjugate to

γ_2. Then there exists a Möbius transformation of $\mathbb{H}$, $\gamma(z) = (az+b)/(cz+d)$, such that $\gamma\gamma_1(z) = \gamma_2\gamma(z)$. Explicitly:

$$\frac{ak_1 z + b}{ck_1 z + d} = k_2 \left(\frac{az+b}{cz+d} \right).$$

Multiplying out and equating coefficients gives

$$ack_1 = ack_1k_2, \ adk_1 + bc = k_2ad + k_1k_2bc, \ bd = k_2bd.$$

As $k_2 \neq 1$ the third equation implies that $bd = 0$.

Case 1: $b = 0$. If $b = 0$ then the second equation implies that $adk_1 = adk_2$. So either $k_1 = k_2$ or $ad = 0$. If $ad = 0$ then, as $b = 0$, we have $ad - bc = 0$ so γ is not a Möbius transformation of $\mathbb{H}$. Hence $k_1 = k_2$.

Case 2: $d = 0$. If $d = 0$ then $bc = bck_1k_2$. So either $k_1k_2 = 1$ or $bc = 0$. If $bc = 0$ then, as $d = 0$, we have $ad - bc = 0$ so γ is not a Möbius transformation of $\mathbb{H}$. Hence $k_1k_2 = 1$.

Here is a sketch of an alternative method. If $\gamma_1(z) = k_1z$ and $\gamma_2(z) = k_2z$ are conjugate then they have the same trace. The trace of γ_1 is seen in Exercise 11.2 below to be $(\sqrt{k_1} + 1/\sqrt{k_1})^2$, and the trace of γ_2 is $(\sqrt{k_2} + 1/\sqrt{k_2})^2$. Equating these shows (after some manipulation) that $k_1 = k_2$ or $k_1 = 1/k_2$.

Solution 11.2

Let γ be hyperbolic. Then γ is conjugate to a dilation $z \mapsto kz$. Writing this dilation in a normalised form

$$z \mapsto \frac{\frac{k}{\sqrt{k}} z}{\frac{1}{\sqrt{k}}}$$

we see that

$$\tau(\gamma) = \left(\sqrt{k} + \frac{1}{\sqrt{k}} \right)^2.$$

Solution 11.3

Let γ be an elliptic Möbius transformation. Then γ is conjugate (as a Möbius transformation of $\mathbb{D}$) to the rotation of $\mathbb{D}$ by θ, i.e. γ is conjugate to $z \mapsto e^{i\theta}z$. Writing this transformation in a normalised form we have

$$z \mapsto \frac{e^{i\theta/2}z}{e^{-i\theta/2}},$$

which has trace

$$(e^{i\theta/2} + e^{-i\theta/2})^2 = 4\cos^2(\theta/2).$$

Hence $\tau(\gamma) = 4\cos^2(\theta/2)$.

Solution 12.1

Fix $q > 0$ and let

$$\Gamma_q = \left\{ \gamma(z) = \frac{az+b}{cz+d} \mid a, b, c, d \in \mathbb{Z}, \ b, c \text{ are divisible by } q \right\}.$$

First note that $\mathrm{id} \in \Gamma_q$ (take $a = d = 1, b = c = 0$).

Let $\gamma_1 = (a_1 z + b_1)/(c_1 z + d_2), \gamma_2 = (a_2 z + b_2)/(c_2 z + d_2) \in \Gamma_q$. Then

$$\gamma_1\gamma_2(z) = \frac{(a_1a_2 + b_1c_2)z + (a_1b_2 + b_1d_2)}{(c_1a_2 + d_1c_2)z + (c_1b_2 + d_1d_2)}.$$

Now q divides b_1, b_2, c_1, c_2. Hence q divides $a_1b_2 + b_1d_2$ and $c_1a_2 + d_1c_2$. Hence $\gamma_1\gamma_2 \in \Gamma_q$.

If $\gamma(z) = (az+b)/(cz+d) \in \Gamma_q$ then $\gamma^{-1}(z) = (dz-b)/(-cz+a)$. Hence $\gamma^{-1} \in \Gamma_q$.

Hence Γ_q is a subgroup of Möb($\mathbb{H}$).

Solution 12.2

The group generated by $\gamma_1(z) = z+1$ and $\gamma_2(z) = kz$ ($k \neq 1$) is not a Fuchsian group. Consider the orbit $\Gamma(i)$ of i. First assume that $k > 1$. Then observe that

$$\gamma_2^{-n}\gamma_1^m\gamma_2^n(i) = \gamma_2^{-n}\gamma_1^m(k^n i) = \gamma_2^{-n}(k^n i + m) = i + m/k^n.$$

By choosing n arbitrarily large we see that m/k^n is arbitrarily close to, but not equal to, 0. Hence i is not an isolated point of the orbit $\Gamma(i)$. Hence $\Gamma(i)$ is not discrete. By Proposition 12.5.1, Γ is not a Fuchsian group.

The case where $0 < k < 1$ is similar, but with $\gamma_2^{-n}\gamma_1^m\gamma_2^n$ replaced by $\gamma_2^n\gamma_1^m\gamma_2^{-n}$

Solution 13.1

See Figure 25.6.

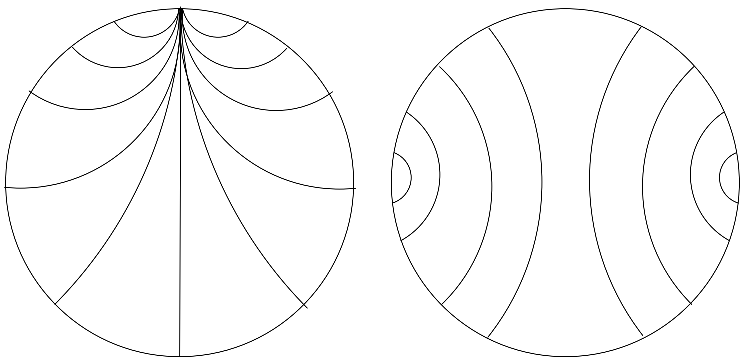

Figure 25.6: Solution to Exercise ex:examplesoftwotessellations.

Solution 14.1

(Not examinable—included for interest only!)

Recall that a subset $C \subset \mathbb{H}$ is *convex* if: $\forall z, w \in C, [z,w] \subset C$; that is, the geodesic segment between any two points of C lies inside C.

Let us first show that a half-plane is convex. We first show that the half-plane $H_0 = \{z \in \mathbb{H} \mid \mathrm{Re}(z) > 0\}$ is convex; in fact this is obvious by drawing a picture. Now let H be any half-plane; we have to show that H is convex. Recall that H is defined by a geodesic ℓ of $\mathbb{H}$ and that the group of Möbius transformations of $\mathbb{H}$ acts transitively on geodesics. Hence we can find a Möbius transformation γ of $\mathbb{H}$ that maps the imaginary axis to ℓ. Hence γ maps either H_0 or $\{z \in \mathbb{H} \mid \mathrm{Re}(z) < 0\}$ to H. In the latter case we can first apply the isometry $z \mapsto -\bar{z}$ so that H_0 is mapped by an isometry to H. As isometries map geodesic segments to geodesic segments, we see that H is convex.

Finally, let $D = \cap H_i$ be an intersection of half-planes. Let $z, w \in D$. Then $z, w \in H_i$ for each i. As H_i is convex, the geodesic segment $[z, w] \subset H_i$ for each i. Hence $[z, w] \subset D$ so that D is convex.

Solution 14.2

(i) By Proposition 14.3.1, $z \in \mathbb{H}$ is on the perpendicular bisector of $[z_1, z_2]$ if and only if $d_{\mathbb{H}}(z, z_1) = d_{\mathbb{H}}(z, z_2)$. Note that

$$\begin{aligned} d_{\mathbb{H}}(z, z_1) = d_{\mathbb{H}}(z, z_2) \quad &\Leftrightarrow \quad \cosh d_{\mathbb{H}}(z, z_1) = \cosh d_{\mathbb{H}}(z, z_2) \\ &\Leftrightarrow \quad 1 + \frac{|z - z_1|^2}{2y_1 \operatorname{Im}(z)} = 1 + \frac{|z - z_2|^2}{2y_2 \operatorname{Im}(z)} \\ &\Leftrightarrow \quad y_2|z - z_1|^2 = y_1|z - z_2|^2. \end{aligned}$$

(ii) Let $z = x + iy$. Then z is on the perpendicular bisector of $1 + 2i$ and $(6 + 8i)/5$ precisely when

$$\frac{8}{5}|(x + iy) - (1 + 2i)|^2 = 2|(x + iy) - \left(\frac{6}{5} + \frac{8i}{5}\right)|^2,$$

i.e.

$$4\left((x - 1)^2 + (y - 2)^2\right) = 5\left(\left(x - \frac{6}{5}\right)^2 + \left(y - \frac{8}{5}\right)^2\right).$$

Expanding this out and collecting like terms gives

$$x^2 - 4x + y^2 = 0$$

and completing the square gives

$$(x - 2)^2 + y^2 = 4 = 2^2.$$

Hence the perpendicular bisector is the semi-circle with centre $(2, 0)$ and radius 2.

Solution 15.1

Let $\Gamma = \{\gamma_n \mid \gamma_n(z) = 2^n z\}$. Let $p = i$ and note that $\gamma_n(p) = 2^n i \neq p$ unless $n = 0$. For each n, $[p, \gamma_n(p)]$ is the arc of imaginary axis from i to $2^n i$. Suppose first that $n > 0$. Recalling that for $a < b$ we have $d_{\mathbb{H}}(ai, bi) = \log b/a$ it is easy to see that the midpoint of $[i, 2^n i]$ is at $2^{n/2} i$. Hence $L_p(\gamma_n)$ is the semicircle of radius $2^{n/2}$ centred at the origin and

$$H_p(\gamma_n) = \{z \in \mathbb{H} \mid |z| < 2^{n/2}\}.$$

For $n < 0$ one sees that

$$H_p(\gamma_n) = \{z \in \mathbb{H} \mid |z| > 2^{n/2}\}.$$

Hence

$$\begin{aligned} D(p) &= \bigcap_{\gamma_n \in \Gamma \setminus \{\mathrm{Id}\}} H_p(\gamma_n) \\ &= \{z \in \mathbb{H} \mid 1/\sqrt{2} < |z| < \sqrt{2}\}. \end{aligned}$$

Solution 16.1
Let $p = i$ and let $\gamma_n(z) = 2^n z$. There are two sides:

$$\begin{aligned} s_1 &= \{z \in \mathbb{C} \mid |z| = 1/\sqrt{2}\}, \\ s_2 &= \{z \in \mathbb{C} \mid |z| = \sqrt{2}\}. \end{aligned}$$

The side s_1 is the perpendicular bisector of $[p, \gamma_{-1}(p)]$. Hence γ_{s_1}, the side-pairing transformation associated to the side s_1, is

$$\gamma_{s_1}(z) = (\gamma_{-1})^{-1}(z) = 2z$$

and pairs side s_1 to side s_2. Hence $\gamma_{s_2}(z) = \gamma_{s_1}^{-1}(z) = z/2$.

Solution 17.1

(i) This follows by observing that running the algorithm starting at $(v, *s)$ is the same as running the algorithm for (v, s) backwards.

(ii) Starting the algorithm at (v_i, s_i) is the same as starting from the i^{th} stage of the algorithm started at (v_0, s_0).

Solution 17.2
Suppose the vertices in the elliptic cycle are labelled so that the elliptic vertex cycle is

$$v_0 \to v_2 \to \cdots \to v_{n-1}$$

and the side-pairing transformations are labelled so that the elliptic cycle is given by

$$\gamma_{v_0,s_0} = \gamma_n \gamma_{n-1} \cdots \gamma_1.$$

Suppose that γ_{v_0,s_0} has order $m > 0$.
Now consider the pair (v_i, s_i). Then the elliptic cycle is given by

$$\begin{aligned} \gamma_{v_i,s_i} &= \gamma_i \gamma_{i-1} \cdots \gamma_1 \gamma_n \cdots \gamma_{i+1} \\ &= (\gamma_i \cdots \gamma_1)\gamma_{v_0,s_0}(\gamma_i \cdots \gamma_1)^{-1}. \end{aligned}$$

Then

$$\begin{aligned} \gamma_{v_i,s_i}^m &= (\gamma_i \cdots \gamma_1)\gamma_{v_0,s_0}(\gamma_i \cdots \gamma_1)^{-1}(\gamma_i \cdots \gamma_1)\gamma_{v_0,s_0}(\gamma_i \cdots \gamma_1)^{-1} \\ &\quad \cdots (\gamma_i \cdots \gamma_1)\gamma_{v_0,s_0}(\gamma_i \cdots \gamma_1)^{-1} \\ &= (\gamma_i \cdots \gamma_1)\gamma_{v_0,s_0}^m(\gamma_i \cdots \gamma_1)^{-1} \\ &= (\gamma_i \cdots \gamma_1)(\gamma_i \cdots \gamma_1)^{-1} \\ &= \mathrm{Id}. \end{aligned}$$

Hence γ_{v_i,s_i} has order m.

Solution 18.1
Let $\Gamma = \langle a, b \mid a^4 = b^2 = (ab)^2 = e\rangle$. First note that $e = a^4 = a^3 a$ and $e = b^2 = bb$ so that $a^{-1} = a^3$ and $b^{-1} = b$. Now $e = (ab)^2 = abab$ and multiplying on the left first by a^{-1}

and then b^{-1} gives that $ab = ba^3$. (Note that one cannot write $(ab)^2 \neq a^2b^2$.) From this it follows that

$$a^2b = a(ab) = a(ba^3) = (ab)a^3 = ba^3a^3 = ba^6 = ba^2a^4 = ba^2$$

and similarly

$$a^3b = a(a^2b) = a(ba^2) = (ab)a^3 = ba^2a^3 = ba^5 = ba.$$

Now let $w \in \Gamma$ be a finite word in Γ. Then

$$w = a^{n_1}b^{m_1} \cdots a^{n_k}b^{m_k}$$

for suitable integers n_j, m_j. Using the relations $a^4 = b^2 = e$ we can assume that $n_j \in \{0, 1, 2, 3\}$ and $m_j \in \{0, 1\}$. Using the relations we deduced above that $ab = ba^3$, $a^2b = ba^2$ and $a^3b = ba$, we can move all of the as to the left and all of the bs to the right to see that we can write $w = a^nb^m$ for suitable integers n, m. Again, as $a^4 = b^2 = e$ we can assume that $n \in \{0, 1, 2, 3\}$ and $m \in \{0, 1\}$. Hence there are exactly 8 elements in Γ.

Solution 19.1
Label the sides and vertices of the quadrilateral as in Figure 25.7. Then

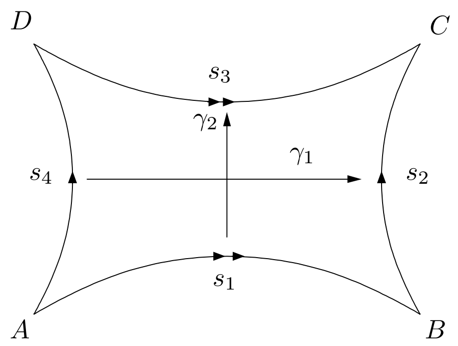

Figure 25.7: A hyperbolic quadrilateral.

$$\begin{array}{lllll}
\begin{pmatrix} A \\ s_1 \end{pmatrix} & \xrightarrow{\gamma_2} & \begin{pmatrix} D \\ s_3 \end{pmatrix} & \xrightarrow{*} & \begin{pmatrix} D \\ s_4 \end{pmatrix} \\
& \xrightarrow{\gamma_1} & \begin{pmatrix} C \\ s_2 \end{pmatrix} & \xrightarrow{*} & \begin{pmatrix} C \\ s_3 \end{pmatrix} \\
& \xrightarrow{\gamma_2^{-1}} & \begin{pmatrix} B \\ s_1 \end{pmatrix} & \xrightarrow{*} & \begin{pmatrix} B \\ s_2 \end{pmatrix} \\
& \xrightarrow{\gamma_1^{-1}} & \begin{pmatrix} A \\ s_4 \end{pmatrix} & \xrightarrow{*} & \begin{pmatrix} A \\ s_1 \end{pmatrix}.
\end{array}$$

Hence the elliptic cycle is $A \to D \to C \to B$ and the corresponding elliptic cycle transformation is $\gamma_1^{-1}\gamma_2^{-1}\gamma_1\gamma_2$.

If we let $\angle A$ denote the internal angle at A, with similar notation for the other vertices, then the angle sum is $\text{sum}(A) = \angle A + \angle B + \angle C + \angle D$.

By Poincaré's Theorem, γ_1, γ_2 generate a Fuchsian group if and only if

$$m(\angle A + \angle B + \angle C + \angle D) = 2\pi$$

for some integer $m \geq 1$.

Solution 20.1
Label the sides as in Figure 25.8. Then

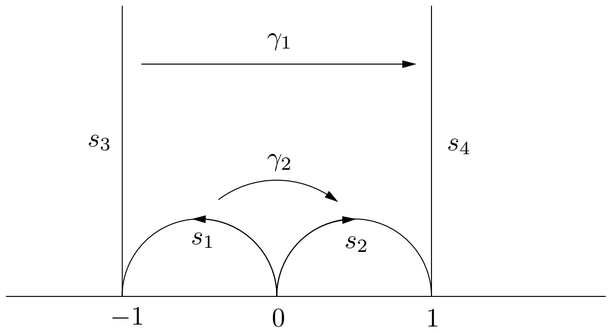

Figure 25.8: A fundamental domain for the free group on 2 generators.

$$\begin{array}{cccc} \begin{pmatrix} -1 \\ s_1 \end{pmatrix} & \stackrel{\gamma_2}{\to} & \begin{pmatrix} 1 \\ s_2 \end{pmatrix} \stackrel{*}{\to} \begin{pmatrix} 1 \\ s_4 \end{pmatrix} \\ & \stackrel{\gamma_1^{-1}}{\to} & \begin{pmatrix} -1 \\ s_3 \end{pmatrix} \stackrel{*}{\to} \begin{pmatrix} -1 \\ s_1 \end{pmatrix}, \end{array}$$

and

$$\begin{pmatrix} \infty \\ s_3 \end{pmatrix} \stackrel{\gamma_1}{\to} \begin{pmatrix} \infty \\ s_4 \end{pmatrix} \stackrel{*}{\to} \begin{pmatrix} \infty \\ s_3 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 \\ s_1 \end{pmatrix} \stackrel{\gamma_2}{\to} \begin{pmatrix} 0 \\ s_2 \end{pmatrix} \stackrel{*}{\to} \begin{pmatrix} 0 \\ s_1 \end{pmatrix}.$$

Hence there are 3 vertex cycles: $-1 \to 1$, ∞ and 0. The corresponding parabolic cycles are: $\gamma_1^{-1}\gamma_2$, γ_1 and γ_2, respectively.

$$\begin{array}{rll} -1 \to 1 & \text{with corresponding parabolic cycle transformation} & \gamma_1^{-1}\gamma_2, \\ \infty & \text{with corresponding parabolic cycle transformation} & \gamma_1, \\ 0 & \text{with corresponding parabolic cycle transformation} & \gamma_2. \end{array}$$

Clearly γ_1 is parabolic (it is a translation and so has a single fixed point at ∞). The map γ_2 is parabolic; it is normalised and has trace $\tau(\gamma_2) = (1+1)^2 = 4$. Finally, the map $\gamma_1^{-1}\gamma_2$ is given by:

$$\gamma_1^{-1}\gamma_2(z) = \gamma_1^{-1}\left(\frac{z}{2z+1}\right) = \frac{z}{2z+1} - 2 = \frac{-3z-2}{2z+1}$$

which is normalised; hence $\tau(\gamma_1^{-1}\gamma_2) = (-3+1)^2 = 4$ so that $\gamma_1^{-1}\gamma_2$ is parabolic.

By Poincaré's Theorem, as all parabolic cycle transformations are parabolic (and there are no elliptic cycles), the group Γ generated by γ_1, γ_2 is a Fuchsian group.

As there are no elliptic cycles, there are no relations. Hence the group is isomorphic to $\langle a, b \rangle$ (take $a = \gamma_1, b = \gamma_2$), which is the free group on 2 generators.

Solution 20.2

(i) The side-pairing transformation γ_1 is a translation that clearly maps the side $\mathrm{Re}(z) = -(1+\sqrt{2}/2)$ to the side $\mathrm{Re}(z) = 1+\sqrt{2}/2$. Hence γ_1 is a side-pairing transformation.

Recall that through any two points of $\mathbb{H}\cup\partial\mathbb{H}$ there exists a unique geodesic. The map γ_2 maps the point $i\sqrt{2}/2$ to itself and the point $-(1+\sqrt{2}/2)$ to $1+\sqrt{2}/2$. Hence γ_2 maps the arc of geodesic $[A,B]$ to $[C,B]$. Hence γ_2 is a side-pairing transformation.

(ii) Let s_1 denote the side $[B,A]$, s_2 denote the side $[B,C]$, s_3 denote the side $[C,\infty]$ and s_4 denote the side $[A,\infty]$.

Now
$$\begin{pmatrix} B \\ s_1 \end{pmatrix} \xrightarrow{\gamma_2} \begin{pmatrix} B \\ s_2 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} B \\ s_1 \end{pmatrix}.$$
Hence we have an elliptic cycle $\mathcal{E} = B$ with elliptic cycle transformation γ_2 and corresponding angle sum $\mathrm{sum}(\mathcal{E}) = \angle B = \pi/2$. As $4\pi/2 = 2\pi$, the elliptic cycle condition holds with $m_{\mathcal{E}} = 4$.

Now consider the following parabolic cycle:
$$\begin{pmatrix} \infty \\ s_4 \end{pmatrix} \xrightarrow{\gamma_1} \begin{pmatrix} \infty \\ s_3 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} \infty \\ s_4 \end{pmatrix}.$$
Hence we have a parabolic cycle $\mathcal{P}_1 = \infty$ with parabolic cycle transformation γ_1. As γ_1 is a translation, it must be parabolic (recall that all parabolic Möbius transformations of $\mathbb{H}$ are conjugate to a translation). Hence the parabolic cycle condition holds.

Finally, we have the parabolic cycle:
$$\begin{aligned} \begin{pmatrix} A \\ s_4 \end{pmatrix} &\xrightarrow{\gamma_1} \begin{pmatrix} C \\ s_3 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} C \\ s_2 \end{pmatrix} \\ &\xrightarrow{\gamma_2^{-1}} \begin{pmatrix} A \\ s_1 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} A \\ s_4 \end{pmatrix}. \end{aligned}$$
Hence we have a parabolic cycle $\mathcal{P}_2 = A \to C$ with parabolic cycle transformation: $\gamma_2^{-1}\gamma_1$. Now $\gamma_2^{-1}\gamma_1$ has the matrix
$$\begin{pmatrix} \sqrt{2}/2 & 1/2 \\ -1 & \sqrt{2}/2 \end{pmatrix} \begin{pmatrix} 1 & 2+\sqrt{2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}+1 \\ -1 & -2-\sqrt{2}/2 \end{pmatrix}$$
which is normalised. Hence the trace of $\gamma_2^{-1}\gamma_1$ is
$$\left(\frac{\sqrt{2}}{2} - 2 - \frac{\sqrt{2}}{2}\right)^2 = 4.$$
Using the fact that a Möbius transformation is parabolic if and only if it has trace 4, we see that $\gamma_2^{-1}\gamma_1$ is parabolic. Hence the parabolic cycle condition holds.

By Poincaré's Theorem, γ_1 and γ_2 generate a Fuchsian group. In terms of generators and relations, it is given by
$$\langle a, b \mid b^4 = e \rangle.$$
(Here we take $a = \gamma_1, b = \gamma_2$. The relation b^4 comes from the fact that the elliptic cycle $\mathcal{E} = B$ has elliptic cycle transformation $\gamma_{\mathcal{E}} = \gamma_2$ with angle sum $\pi/2$. Hence $m_{\mathcal{E}} = 4$. The relation $\gamma_{\mathcal{E}}^{m_{\mathcal{E}}}$ is then b^4.)

Solution 21.1

(i) First note that one side of the polygon is paired with itself. Introduce a new vertex at the mid-point of this side, introducing two new sides each of which is paired with the other. Label the polygon as in Figure 25.9.

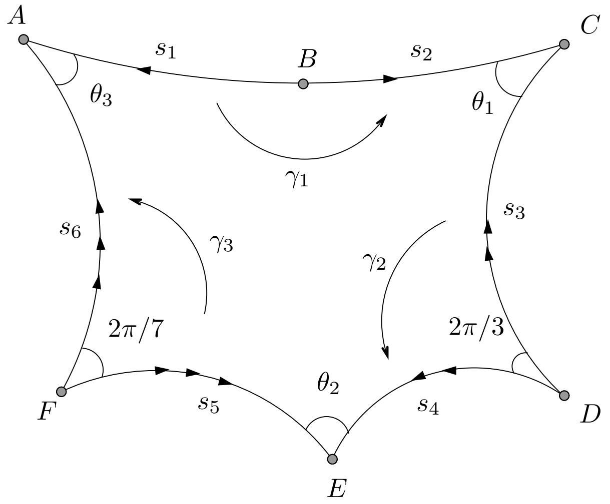

Figure 25.9: Labelling the hyperbolic polygon, remembering to add an extra vertex to the side that is paired with itself.

Then

$$\begin{pmatrix} B \\ s_1 \end{pmatrix} \overset{\gamma_1}{\to} \begin{pmatrix} B \\ s_2 \end{pmatrix} \overset{*}{\to} \begin{pmatrix} B \\ s_1 \end{pmatrix}.$$

This gives an elliptic cycle $\mathcal{E}_1 = B$ with elliptic cycle transformation γ_1 and angle sum $\mathrm{sum}(\mathcal{E}_1) = \pi$. Hence the elliptic cycle condition holds with $m_1 = 2$.

We also have

$$\begin{pmatrix} D \\ s_3 \end{pmatrix} \overset{\gamma_2}{\to} \begin{pmatrix} D \\ s_4 \end{pmatrix} \overset{*}{\to} \begin{pmatrix} D \\ s_3 \end{pmatrix}.$$

This gives an elliptic cycle $\mathcal{E}_2 = D$ with elliptic cycle transformation γ_2 and angle sum $\mathrm{sum}(\mathcal{E}_2) = 2\pi/3$. Hence the elliptic cycle condition holds with $m_1 = 3$.

Also

$$\begin{pmatrix} F \\ s_5 \end{pmatrix} \overset{\gamma_3}{\to} \begin{pmatrix} F \\ s_6 \end{pmatrix} \overset{*}{\to} \begin{pmatrix} F \\ s_5 \end{pmatrix}.$$

This gives an elliptic cycle $\mathcal{E}_3 = F$ with elliptic cycle transformation γ_3 and angle sum $\mathrm{sum}(\mathcal{E}_3) = 2\pi/7$. Hence the elliptic cycle condition holds with $m_1 = 7$.

Finally

$$\begin{aligned} \begin{pmatrix} A \\ s_1 \end{pmatrix} &\overset{\gamma_1}{\to} \begin{pmatrix} C \\ s_2 \end{pmatrix} \overset{*}{\to} \begin{pmatrix} C \\ s_3 \end{pmatrix} \\ &\overset{\gamma_2}{\to} \begin{pmatrix} E \\ s_4 \end{pmatrix} \overset{*}{\to} \begin{pmatrix} E \\ s_5 \end{pmatrix} \\ &\overset{\gamma_3}{\to} \begin{pmatrix} A \\ s_6 \end{pmatrix} \overset{*}{\to} \begin{pmatrix} A \\ s_1 \end{pmatrix}. \end{aligned}$$

This gives an elliptic cycle $\mathcal{E}_4 = A \to C \to E$ with elliptic cycle transformation $\gamma_3\gamma_2\gamma_1$. The angle sum is $\mathrm{sum}(\mathcal{E}_3) = \theta_1 + \theta_2 + \theta_3 = 2\pi$. Hence the elliptic cycle condition holds with $m_4 = 1$. Hence $\mathcal{E}_4$ is an accidental cycle.

(ii) By Poincaré's Theorem, $\gamma_1, \gamma_2, \gamma_3$ generate a Fuchsian group Γ. In terms of generators and relations we can write

$$\Gamma = \langle a, b, c \mid a^2 = b^3 = c^7 = abc = e \rangle.$$

(iii) To calculate the genus of $\mathbb{H}/\Gamma$ we use Euler's formula $2-2g = V - E + F$. Recall that each elliptic cycle on the polygon glues together to give one vertex on a triangulation of $\mathbb{H}/\Gamma$. As there are 4 elliptic cycles we have $V = 4$. Each pair of paired sides in the polygon glue together to give one edge on a triangulation of $\mathbb{H}/\Gamma$. As there are 6 sides in the polygon, there are $E = 6/2 = 3$ edges in the trinagulation of $\mathbb{H}/\Gamma$. As we are only using 1 polygon, there is $F = 1$ face of the triangulation of $\mathbb{H}/\Gamma$. Hence $2 - 2g = V - E + F = 4 - 3 + 1 = 2$, so that $g = 0$.

As the orders of the non-accidental elliptic cycles are $2, 3, 7$, we see that $\mathrm{sig}(\Gamma) = (0; 2, 3, 7)$.

Solution 21.2

From Exercise 7.3, we know that there exists a regular hyperbolic n-gon with internal angle θ provided $(n-2)\pi - 8\theta > 0$. When $n = 8$, this rearranges to $\theta \in [0, 3\pi/4)$.

Label the vertices of the octagon as indicated in Figure 25.10.

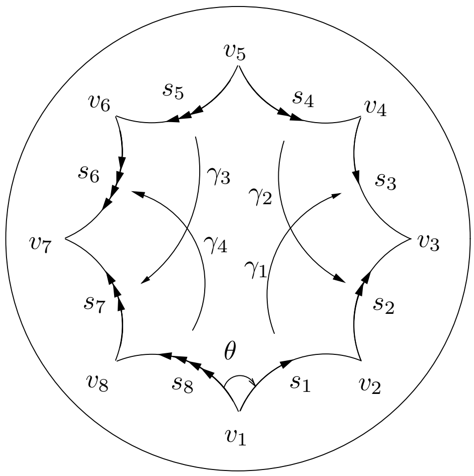

Figure 25.10: See the solution to Exercise 21.2.

We have

$$\begin{pmatrix} v_1 \\ s_1 \end{pmatrix} \xrightarrow{\gamma_1} \begin{pmatrix} v_4 \\ s_3 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} v_4 \\ s_4 \end{pmatrix}$$
$$\xrightarrow{\gamma_2} \begin{pmatrix} v_3 \\ s_2 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} v_3 \\ s_3 \end{pmatrix}$$
$$\xrightarrow{\gamma_1^{-1}} \begin{pmatrix} v_2 \\ s_1 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} v_2 \\ s_2 \end{pmatrix}$$

$$\begin{aligned}
&\xrightarrow{\gamma_2^{-1}} \begin{pmatrix} v_5 \\ s_4 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} v_5 \\ s_5 \end{pmatrix}\\
&\xrightarrow{\gamma_3} \begin{pmatrix} v_8 \\ s_7 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} v_8 \\ s_8 \end{pmatrix}\\
&\xrightarrow{\gamma_4} \begin{pmatrix} v_7 \\ s_6 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} v_7 \\ s_7 \end{pmatrix}\\
&\xrightarrow{\gamma_3^{-1}} \begin{pmatrix} v_6 \\ s_5 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} v_6 \\ s_6 \end{pmatrix}\\
&\xrightarrow{\gamma_4^{-1}} \begin{pmatrix} v_1 \\ s_8 \end{pmatrix} \xrightarrow{*} \begin{pmatrix} v_1 \\ s_1 \end{pmatrix}.
\end{aligned}$$

Thus there is just one elliptic cycle:

$$\mathcal{E} = v_1 \to v_4 \to v_3 \to v_2 \to v_5 \to v_8 \to v_7 \to v_6.$$

with associated elliptic cycle transformation:

$$\gamma_4^{-1}\gamma_3^{-1}\gamma_4\gamma_3\gamma_2^{-1}\gamma_1^{-1}\gamma_2\gamma_1$$

As the internal angle at each vertex is θ, the angle sum is 8θ Hence the elliptic cycle condition holds whenever there exists an integer $m = m_{\mathcal{E}}$ such that $8m\theta = 2\pi$, i.e. whenever $\theta = \pi/4m$ for some integer m. When $m = 1$ this is an accidental cycle.

Let θ be such that $\theta = \pi/4m$ for some integer m. Then by Poincaré's Theorem, the group $\Gamma_{\pi/4m}$ generated by the side-pairing transformations $\gamma_1, \ldots, \gamma_4$ generate a Fuchsian group. Moreover, we can write this group in terms of generators and relations as follows:

$$\Gamma_{\pi/4m} = \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 \mid (\gamma_4^{-1}\gamma_3^{-1}\gamma_4\gamma_3\gamma_2^{-1}\gamma_1^{-1}\gamma_2\gamma_1)^m = e\rangle.$$

The quotient space $\mathbb{H}/\Gamma_{\pi/4m}$ is a torus of genus 2. When $m = 1$, $\mathrm{sig}(\Gamma_{\pi/4}) = (2, -)$ and $\mathbb{H}/\Gamma_{\pi/4}$ has no marked points. When $m \geq 2$ then $\mathrm{sig}(\Gamma_{\pi/4}) = (2, m)$ and $\mathbb{H}/\Gamma_{\pi/4m}$ has one marked point of order m.

Solution 21.3

(i) Consider the Dirichlet polygon and side-pairing transformations for the modular group that we constructed in Lecture 15. See Figure 25.11. The sides s_1 and s_2 are paired. This gives one cusp at the point ∞.

There are two elliptic cycles: $A \to B$ (which has an angle sum of $2\pi/3$), and i (which has an angle sum of π). Hence when we glue together the vertices A and B we get a marked point of order 3, and the vertex i gives a marked point of order 2.

We do not get any 'holes' when we glue together the sides. Hence we have genus 0.

Thus the modular group has signature $(0; 2, 3; 1)$.

(ii) By Proposition 13.2.1 it is sufficient to prove that the formula holds for a Dirichlet polygon D. Suppose that D has n vertices (hence n sides).

We use the Gauss-Bonnet Theorem (Theorem 7.2.1). By Proposition 17.3.1, the angle sum along the j^{th} non-accidental elliptic cycle $\mathcal{E}_j$ is

$$\mathrm{sum}(\mathcal{E}_j) = \frac{2\pi}{m_j}.$$

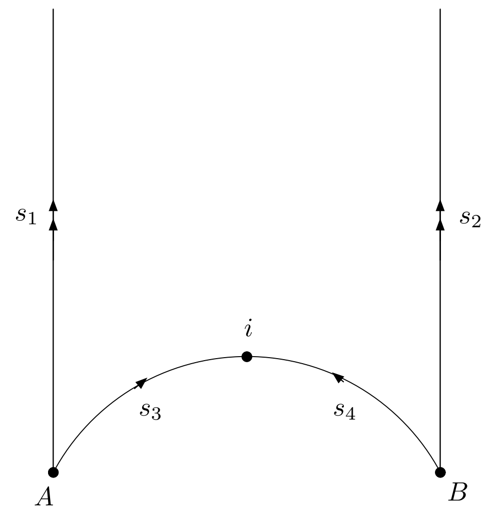

Figure 25.11: A fundamental domain and side-pairing transformations for the modular group.

Hence the sum of the interior angles of vertices on non-accidental elliptic cycles is

$$\sum_{j=1}^{r} \frac{2\pi}{m_j}.$$

Suppose that there are s accidental cycles. (Recall that a cycle is said to be *accidental* if the corresponding elliptic cycle transformation is the identity, and in particular has order 1.) By Proposition 17.3.1, the internal angle sum along an accidental cycle is 2π. Hence the internal angle sum along all accidental cycles is $2\pi s$.

Suppose that there are c parabolic cycles. The angle sum along a parabolic cycle must be zero (the vertices must be on the boundary, and the angle between two geodesics that intersect on the boundary must be zero).

As each vertex belongs to either a non-accidental elliptic cycle, to an accidental cycle or to a parabolic cycle, the sum of all the internal angles of D is given by

$$2\pi \left(\sum_{j=1}^{r} \frac{1}{m_j} + s \right).$$

By the Gauss-Bonnet Theorem, we have

$$\mathrm{Area}_{\mathbb{H}}(D) = (n-2)\pi - 2\pi \left(\sum_{j=1}^{r} \frac{1}{m_j} + s \right). \tag{25.5}$$

Consider now the space $\mathbb{H}/\Gamma$. This is formed by taking D and glueing together paired sides. The vertices along each elliptic cycle, accidental cycle and parabolic cycle are glued together to form a vertex in $\mathbb{H}/\Gamma$. Hence the number of vertices in $\mathbb{H}/\Gamma$ is equal to the number of cycles (elliptic, accidental and parabolic); hence D corresponds to a triangulation of $\mathbb{H}/\Gamma$ with $V = r + s + c$ vertices. As paired sides are glued together, there are $E = n/2$ edges. Finally, as we only need the single polygon D, there is only $F = 1$ face. Hence

$$2 - 2g = \chi(\mathbb{H}/\Gamma) = r + s + c - \frac{n}{2} + 1$$

which rearranges to give

$$n - 2 = 2((r + s + c) - (2 - 2g)). \tag{25.6}$$

Substituting (25.6) into (25.5) we see that

$$\begin{aligned}\mathrm{Area}_{\mathbb{H}}(D) &= 2\pi\left(r + s + c - (2 - 2g) - \sum_{j=1}^{r} \frac{1}{m_j} - s\right)\\ &= 2\pi\left((2g - 2) + \sum_{j=1}^{r}\left(1 - \frac{1}{m_j}\right) + c\right).\end{aligned}$$

(iii) We must show that

$$(2g - 2) + \sum_{j=1}^{r}\left(1 - \frac{1}{m_j}\right) + c \geq \frac{1}{6}. \tag{25.7}$$

We assume that $c \geq 1$.

If $g \geq 1$ then $2g - 2 + c \geq 1 > 1/6$, so that (25.7) holds. So it remains to check the cases when $g = 0$.

If $g = 0$ and $c \geq 2$ then $2g - 2 + c \geq 0$. As $1 - 1/m_j \geq 1/2$, it follows that the left-hand side of (25.7) is at least $1/2$. Hence (25.7) holds. So it remains to check the cases when $g = 0$ and $c = 1$.

If $g = 0$ and $c = 1$ then $2g - 2 + c = -1$. As $m_j \geq 2$, we see that $1 - 1/m_j \geq 1/2$. Hence if $r \geq 3$ then the left-hand side of (25.7) is at least $1/2$. Hence (25.7) holds. It remains to check that case when $g = 0$, $c = 1$ and $r = 2$.

In this case, it remains to check that

$$s(k, l) = 1 - \frac{1}{k} - \frac{1}{l} \geq \frac{1}{6}$$

(letting $k = m_1, l = m_2$). We may assume that $k \leq l$. Now $s(3, 3) = 1/3 > 1/6$ and $s(3, l) \geq 1/3$ for $l \geq 3$. Hence we may assume that $k = 2$. Then $s(2, 2) = 0$, $s(2, 3) = 1/6$ and $s(2, l) > 1/6$. Hence the minimum is achieved for $k = 2, l = 3$.

Hence the minimum is achieved for a Fuchsian group with signature $(0; 2, 3; 1)$. By part (i), this is the signature of the modular group.