24. All of the exercises

§24.1 Introduction

The exercises are scattered throughout the notes where they are relevant to the material being discussed. For convenience, all of the exercises are given below. The numbering convention is that Exercise $n.m$ is the mth exercise in lecture n. Thus, once we've done lecture n in class, you will be able to do all the exercises numbered $n.m$.

Particularly unimportant exercises, notably those that are there purely for completeness (such as proving that a given definition makes sense, or illustrating a minor point from the lectures) are labelled $\flat$.

§24.2 The exercises

Exercise 1.1$\flat$
Let R_θ denote the 2×2 matrix that rotates $\mathbb{R}^2$ clockwise about the origin through angle $\theta \in [0, 2\pi)$. Thus R_θ has matrix

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Let $a = (a_1, a_2) \in \mathbb{R}^2$. Define the transformation

$$T_{\theta,a} : \mathbb{R}^2 \to \mathbb{R}^2$$

by

$$T_{\theta,a}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a_1 \\ a_2 \end{pmatrix};$$

thus $T_{\theta,a}$ first rotates the point (x, y) about the origin through an angle θ and then translates by the vector a.

Let $G = \{T_{\theta,a} \mid \theta \in [0, 2\pi), a \in \mathbb{R}^2\}$.

(i) Let $\theta, \phi \in [0, 2\pi)$ and let $a, b \in \mathbb{R}^2$. Find an expression for the composition $T_{\theta,a} \circ T_{\phi,b}$. Hence show that G is a group under composition of maps (i.e. show that this product is (a) well-defined (i.e. the composition of two elements of G gives another element of G), (b) associative (hint: you already know that composition of functions is associative), (c) that there is an identity element, and (d) that inverses exist).

(ii) Show that the set of all rotations about the origin is a subgroup of G.

(iii) Show that the set of all translations is a subgroup of G.

One can show that G is actually the group $\mathrm{Isom}^+(\mathbb{R}^2)$ of orientation preserving isometries of $\mathbb{R}^2$ with the Euclidean matrices.

Exercise 2.1
Consider the two parametrisations

$$\begin{aligned} \sigma_1 : [0,2] \to \mathbb{H} \quad &: \quad t \mapsto t + i, \\ \sigma_2 : [1,2] \to \mathbb{H} \quad &: \quad t \mapsto (t^2 - t) + i. \end{aligned}$$

Verify that these two parametrisations define the same path σ.

Let $f(z) = 1/\operatorname{Im}(z)$. Calculate $\int_\sigma f$ using both of these parametrisations.

The point of this exercise is to show that we can often simplify calculating the integral $\int_\sigma f$ of a function f along a path σ by choosing a good parametrisation.

Exercise 2.2
Consider the points i and ai where $0 < a < 1$.

(i) Consider the path σ between i and ai that consists of the arc of imaginary axis between them. Find a parametrisation of this path.

(ii) Show that

$$\mathrm{length}_{\mathbb{H}}(\sigma) = \log 1/a.$$

(Notice that as $a \to 0$, we have that $\log 1/a \to \infty$. This motivates why we call $\mathbb{R} \cup \{\infty\}$ the circle at *infinity*.)

Exercise 2.3
Show that $d_{\mathbb{H}}$ satisfies the *triangle inequality*:

$$d_{\mathbb{H}}(x,z) \leq d_{\mathbb{H}}(x,y) + d_{\mathbb{H}}(y,z), \ \forall \ x,y,z \in \mathbb{H}.$$

That is, the distance between two points is increased if one goes via a third point.

Exercise 3.1
Let L be a straight line in $\mathbb{C}$ with equation (3.3.2). Find a formula for its gradient and intersections with the real and imaginary axes in terms of α, β, γ.

Exercise 3.2
Let C be a circle in $\mathbb{C}$ with equation (3.3.2). Find a formula for the centre and radius of C in terms of α, β, γ.

Exercise 3.3
Let γ be a Möbius transformation of $\mathbb{H}$. Show that γ is a well-defined map $\gamma : \mathbb{H} \to \mathbb{H}$ (that is, if $z \in \mathbb{H}$ then $\gamma(z) \in \mathbb{H}$). Show that γ maps $\mathbb{H}$ to itself bijectively and give an explicit expression for the inverse map.

Exercise 3.4
Prove Proposition 3.5.1 [that the set of Möbius transformations of $\mathbb{H}$ form a group under composition]. (To do this, you must: (i) show that the composition $\gamma_1\gamma_2$ of two Möbius transformations of $\mathbb{H}$ is a Möbius transformation of $\mathbb{H}$, (ii) check associativity (hint: you already know that composition of maps is associative), (iii) show that the identity map $z \mapsto z$ is a Möbius transformation, and (iv) show that if $\gamma \in \mathrm{M\ddot{o}b}(\mathbb{H})$ is a Möbius transformation of $\mathbb{H}$, then γ^{-1} exists and is a Möbius transformation of $\mathbb{H}$.)

Exercise 3.5♭
Show that dilations, translations and the inversion $z \mapsto -1/z$ are indeed Möbius transformations of $\mathbb{H}$ by writing them in the form $z \mapsto (az+b)/(cz+d)$ for suitable $a,b,c,d \in \mathbb{R}$, $ad-bc>0$.

Exercise 3.6
Let A be either an arc of circle in $\mathbb{H}$ or part of a straight line in $\mathbb{H}$. Let $\gamma \in \text{Möb}(\mathbb{H})$. Show that $\gamma(A)$ is also either an arc of circle in $\mathbb{H}$ or part of a straight line in $\mathbb{H}$.

Exercise 4.1♭
Show that if $ad-bc \neq 0$ then γ maps $\partial\mathbb{H}$ to itself bijectively.

Exercise 4.2
Prove the two facts used in the above proof [of Proposition 4.1.2]:

$$\begin{aligned} |\gamma'(z)| &= \frac{ad-bc}{|cz+d|^2}, \\ \text{Im}(\gamma(z)) &= \frac{(ad-bc)}{|cz+d|^2}\,\text{Im}(z). \end{aligned}$$

Exercise 4.3♭
Let $z = x+iy \in \mathbb{H}$ and define $\gamma(z) = -x+iy$. (Note that γ is *not* a Möbius transformation of $\mathbb{H}$.)

(i) Show that γ maps $\mathbb{H}$ to $\mathbb{H}$ bijectively.

(ii) Let $\sigma : [a,b] \to \mathbb{H}$ be a differentiable path. Show that

$$\text{length}_{\mathbb{H}}(\gamma \circ \sigma) = \text{length}_{\mathbb{H}}(\sigma).$$

Hence conclude that γ is an isometry of $\mathbb{H}$.

Exercise 4.4
Let $H_1, H_2 \in \mathcal{H}$. Show that there exists a Möbius transformation γ of $\mathbb{H}$ that maps H_1 to H_2.

Exercise 5.1
Let $H_1, H_2 \in \mathcal{H}$ and let $z_1 \in H_1, z_2 \in H_2$. Show that there exists a Möbius transformation γ of $\mathbb{H}$ such that $\gamma(H_1) = H_2$ and $\gamma(z_1) = z_2$. In particular, conclude that given $z_1, z_2 \in \mathbb{H}$, one can find a Möbius transformation γ of $\mathbb{H}$ such that $\gamma(z_1) = z_2$.

(Hint: you know that there exists $\gamma_1 \in \text{Möb}(\mathbb{H})$ that maps H_1 to the imaginary axis and z_1 to i; similarly you know that there exists $\gamma_2 \in \text{Möb}(\mathbb{H})$ that maps H_2 to the imaginary axis and z_2 to i. What does γ_2^{-1} do?)

Exercise 5.2
For each of the following pairs of points, describe (either by giving an equation in the form $\alpha z\bar{z} + \beta z + \beta\bar{z} + \gamma$, or in words) the geodesic between them:

(i) $-3+4i$, $-3+5i$,

(ii) $-3+4i$, $3+4i$,

(iii) $-3+4i$, $5+12i$.

Exercise 5.3
Prove Proposition 5.5.2 using the following steps. For $z, w \in \mathbb{H}$ let

$$\begin{aligned} \mathrm{LHS}(z,w) &= \cosh d_{\mathbb{H}}(z,w) \\ \mathrm{RHS}(z,w) &= 1 + \frac{|z-w|^2}{2\,\mathrm{Im}(z)\,\mathrm{Im}(w)} \end{aligned}$$

denote the left- and right-hand sides of (5.5.1) [the formula for $\cosh d_{\mathbb{H}}(z,w)$] respectively. We want to show that $\mathrm{LHS}(z,w) = \mathrm{RHS}(z,w)$ for all $z, w \in \mathbb{H}$.

(i) Let $\gamma \in \mathrm{M\ddot{o}b}(\mathbb{H})$ be a Möbius transformation of $\mathbb{H}$. Using the fact that γ is an isometry, prove that
$$\mathrm{LHS}(\gamma(z), \gamma(w)) = \mathrm{LHS}(z,w).$$
Using Exercise 4.2 and Lemma 5.5.1, prove that
$$\mathrm{RHS}(\gamma(z), \gamma(w)) = \mathrm{RHS}(z,w).$$

(ii) Let H denote the geodesic passing through z, w. By Lemma 4.3.1 there exists a Möbius transformation γ of $\mathbb{H}$ that maps H to the imaginary axis. Let $\gamma(z) = ia$ and $\gamma(w) = ib$. Prove, using the fact that $d_{\mathbb{H}}(ia, ib) = \log b/a$ if $a < b$, that for this choice of γ we have
$$\mathrm{LHS}(\gamma(z), \gamma(w)) = \mathrm{RHS}(\gamma(z), \gamma(w)).$$

(iii) Conclude that $\mathrm{LHS}(z,w) = \mathrm{RHS}(z,w)$ for all $z, w \in \mathbb{H}$.

Exercise 5.4
A hyperbolic circle C with centre $z_0 \in \mathbb{H}$ and radius $r > 0$ is defined to be the set of all points of hyperbolic distance r from z_0. Using equation (5.5.1) [the formula for $\cosh d_{\mathbb{H}}(z,w)$], show that a hyperbolic circle is a Euclidean circle (i.e. an ordinary circle) but with a different centre and radius.

Exercise 5.5
Recall that we defined the hyperbolic distance by first defining the hyperbolic length of a piecewise continuously differentiable path σ:

$$\mathrm{length}_{\mathbb{H}}(\sigma) = \int \frac{|\sigma'(t)|}{\mathrm{Im}(\sigma(t))}\,dt = \int_\sigma \frac{1}{\mathrm{Im}(z)}. \tag{24.2.1}$$

We then saw that the Möbius transformations of $\mathbb{H}$ are isometries.

Why did we choose the function $1/\mathrm{Im}\, z$ in (24.2.1)? In fact, one can choose *any* positive function and use it to define the length of a path, and hence the distance between two points. However, the geometry that one gets may be very complicated (for example, there may be many geodesics between two points); alternatively, the geometry may not be very interesting (for example, there may be very few symmetries, i.e. the group of isometries is very small).

The group of Möbius transformations of $\mathbb{H}$ is, as we shall see, a very rich group with lots of interesting structure. The point of this exercise is to show that if we want the Möbius transformations of $\mathbb{H}$ to be isometries then we must define hyperbolic length by (24.2.1).

Let $\rho : \mathbb{H} \to \mathbb{R}$ be a continuous positive function. Define the ρ-length of a path $\sigma : [a,b] \to \mathbb{H}$ to be

$$\mathrm{length}_\rho(\sigma) = \int_\sigma \rho = \int_a^b \rho(\sigma(t))|\sigma'(t)|\,dt.$$

(i) Suppose that length_ρ is invariant under Möbius transformations of $\mathbb{H}$, i.e. if $\gamma \in \mathrm{M\ddot{o}b}(\mathbb{H})$ then $\mathrm{length}_\rho(\gamma \circ \sigma) = \mathrm{length}_\rho(\sigma)$. Prove that

$$\rho(\gamma(z))|\gamma'(z)| = \rho(z). \tag{24.2.2}$$

(Hint: you may use the fact that if f is a continuous function such that $\int_\sigma f = 0$ for every path σ then $f = 0$.)

(ii) By taking $\gamma(z) = z + b$ in (24.2.2), deduce that $\rho(z)$ depends only on the imaginary part of z. Hence we may write ρ as $\rho(y)$ where $z = x + iy$.

(iii) By taking $\gamma(z) = kz$ in (24.2.2), deduce that $\rho(y) = c/y$ for some constant $c > 0$.

Hence, up to a normalising constant c, we see that if we require the Möbius transformations of $\mathbb{H}$ to be isometries, then the distance in $\mathbb{H}$ must be given by the formula we introduced in Lecture 2.

Exercise 5.6

(i) Let C_1 and C_2 be two circles in $\mathbb{R}^2$ with centres c_1, c_2 and radii r_1, r_2, respectively. Suppose C_1 and C_2 intersect. Let θ denote the internal angle at the point of intersection (see Figure 5.6). Show that

$$\cos\theta = \frac{|c_1 - c_2|^2 - (r_1^2 + r_2^2)}{2r_1 r_2}.$$

(ii) Consider the geodesic between -6 and 6 and the geodesic between $4\sqrt{2}$ and $6\sqrt{2}$, as illustrated in Figure 5.6). Both of these geodesics are semi-circles. Find the centre and radius of each semi-circle. Hence use the result in (i) to calculate the angle ϕ.

Exercise 5.7
Suppose that two geodesics intersect as illustrated in Figure 5.6.5. Show that

$$\sin\theta = \frac{2ab}{a^2+b^2}, \quad \cos\theta = \frac{b^2-a^2}{a^2+b^2}.$$

Exercise 6.1$\flat$
Check some of the assertions above, for example:

(i) Show that h maps $\mathbb{H}$ to $\mathbb{D}$ bijectively. Show that h maps $\partial\mathbb{H}$ to $\partial\mathbb{D}$ bijectively.

(ii) Calculate $g(z) = h^{-1}(z)$ and show that

$$g'(z) = \frac{-2}{(-iz+1)^2}, \ \mathrm{Im}(g(z)) = \frac{1-|z|^2}{|-iz+1|^2}.$$

(iii) Mimic the proof of Proposition 4.2.1 to show that the real axis is the unique geodesic joining 0 to $x \in (0,1)$ and that

$$d_{\mathbb{D}}(0,x) = \log\left(\frac{1+x}{1-x}\right).$$

Exercise 6.2
Show that $z \mapsto h\gamma h^{-1}(z)$ is a map of the form

$$z \mapsto \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}, \quad \alpha, \beta \in \mathbb{C}, \ |\alpha|^2 - |\beta|^2 > 0.$$

Exercise 6.3
Check directly that Möb($\mathbb{D}$) is a group under composition.

Exercise 6.4
Show that the geodesics in $\mathbb{D}$ have equations of the form

$$\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \alpha = 0$$

where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}$.

Exercise 6.5
Let $C = \{w \in \mathbb{D} \mid d_{\mathbb{D}}(z_0, w) = r\}$ be a hyperbolic circle in $\mathbb{D}$ with centre z_0 and radius $r > 0$. Calculate the (hyperbolic) circumference and (hyperbolic) area of C.

[Hints: First move C to the origin by using a Möbius transformation of $\mathbb{D}$. Use the formula $d_{\mathbb{D}}(0,x) = \log(1+x)/(1-x)$ to show that this is a Euclidean circle, but with a different radius. To calculate area, use polar co-ordinates.]

Exercise 7.1
Consider the hyperbolic triangle in $\mathbb{H}$ with vertices at 0, $(-1+i\sqrt{3})/2$, $(1+i\sqrt{3})/2$ as illustrated in Figure 7.2.5.

(i) Determine the geodesics that comprise the sides of this triangle.

(ii) Use Exercise 5.6 to calculate the internal angles of this triangle. Hence use the Gauss-Bonnet Theorem to calculate the hyperbolic area of this triangle.

Exercise 7.2
Assuming Theorem 7.2.1 but not Theorem 7.2.2, prove that the area of a hyperbolic quadrilateral with internal angles $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ is given by

$$2\pi - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4).$$

Exercise 7.3
Let $n \geq 3$. By explicit construction, show that there exists a regular n-gon with internal angle equal to α if and only if $\alpha \in [0, (n-2)\pi/n)$.

(Hint: Work in the Poincaré disc $\mathbb{D}$. Let $\omega = e^{2\pi i/n}$ be an n^{th} root of unity. Fix $r \in (0,1)$ and consider the polygon $D(r)$ with vertices at $r, r\omega, r\omega^2, \ldots, r\omega^{n-1}$. This is a regular n-gon (why?). Let $\alpha(r)$ denote the internal angle of $D(r)$. Use the Gauss-Bonnet

Theorem to express the area of $D(r)$ in terms of $\alpha(r)$. Examine what happens as $r \to 0$ and as $r \to 1$. (To examine $\lim_{r\to 0} \mathrm{Area}_{\mathbb{H}} D(r)$, note that $D(r)$ is contained in a hyperbolic circle $C(r)$, and use Exercise 6.5 to calculate $\lim_{r\to 0} \mathrm{Area}_{\mathbb{H}} C(r)$.) You may use without proof the fact that $\alpha(r)$ depends continuously on r.)

In particular, conclude that there there exists a regular n-gon with each internal angle equal to a right-angle whenever $n \geq 5$. This is in contrast with the Euclidean case where, of course, the only regular polygon with each internal angle equal to a right-angle is the square.

Exercise 7.4♭
(This exercise is outside the scope of the course (and therefore not examinable!). However, anybody remotely interested in pure mathematics should get to see what is below at least once!)

A *polyhedron* in $\mathbb{R}^3$ is formed by joining together polygons along their edges. A *platonic solid* is a convex polyhedra where each constituent polygon is a regular n-gon, with k polygons meeting at each vertex.

By mimicking the discussions above, show that there are precisely five platonic solids: the tetrahedron, cube, octahedron, dodecahedron and icosahedron (corresponding to $(n, k) = (3, 3), (4, 3), (3, 4), (5, 3)$ and $(3, 5)$, respectively).

Exercise 8.1
Assuming that $\tan\alpha = \tanh a/\sinh b$, prove that $\sin\alpha = \sinh a/\sinh c$ and $\cos\alpha = \tanh b/\tanh c$.

Exercise 8.2
We now have relationships involving: (i) three angles (the Gauss-Bonnet Theorem), (ii) three sides (Pythagoras' Theorem) and (iii) two sides, one angle. Prove the following relationships between one side and two angles:

$$\cosh a = \cos\alpha \operatorname{cosec}\beta, \;\; \cosh c = \cot\alpha \cot\beta.$$

What are the Euclidean analogues of these identities?

Exercise 8.3
Assuming that $\sin\alpha = 1/\cosh a$, check using standard trig and hyperbolic trig identities that $\cos\alpha = 1/\coth a$ and $\tan\alpha = 1/\sinh a$.

Exercise 8.4
Prove Proposition 8.4.1 in the case when Δ is acute (the obtuse case is a simple modification of the argument, and is left for anybody interested...).

(Hint: label the vertices A, B, C with angle α at vertex A, etc. Drop a perpendicular from vertex B meeting the side $[A, C]$ at, say, D to obtain two right-angled triangles ABD, BCD. Use Pythagoras' Theorem and Proposition 8.2.1 in both of these triangles to obtain an expression for $\sin\alpha$.)

Exercise 9.1
Find the fixed points in $\mathbb{H} \cup \partial\mathbb{H}$ of the following Möbius transformations of $\mathbb{H}$:

$$\gamma_1(z) = \frac{2z+5}{-3z-1}, \;\; \gamma_2(z) = 7z+6, \;\; \gamma_3(z) = -\frac{1}{z}, \;\; \gamma_4(z) = \frac{z}{z+1}.$$

In each case, state whether the map is parabolic, elliptic or hyperbolic.

Exercise 9.2
Normalise the Möbius transformations of $\mathbb{H}$ given in Exercise 9.1.

Exercise 9.3♭

(i) Show that $\mathrm{SL}(2,\mathbb{R})$ is indeed a group (under matrix multiplication). (Recall that G is a group if: (i) if $g, h \in G$ then $gh \in G$, (ii) the identity is in G, (iii) if $g \in G$ then there exists $g^{-1} \in G$ such that $gg^{-1} = g^{-1}g =$ identity.)

(ii) Define the subgroup

$$\mathrm{SL}(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}; ad - bc = 1 \right\}$$

to be the subset of $\mathrm{SL}(2,\mathbb{R})$ where all the entries are integers. Show that $\mathrm{SL}(2,\mathbb{Z})$ is a subgroup of $\mathrm{SL}(2,\mathbb{R})$. (Recall that if G is a group and $H \subset G$ then H is a subgroup if it is itself a group.)

Exercise 10.1

(i) Prove that conjugacy between Möbius transformations of $\mathbb{H}$ is an equivalence relation.

(ii) Show that if γ_1 and γ_2 are conjugate then they have the same number of fixed points. Hence show that if γ_1 is hyperbolic, parabolic or elliptic then γ_2 is hyperbolic, parabolic or elliptic, respectively.

Exercise 10.2
Prove Proposition 10.2.1. (Hint: show that if $A_1, A_2, A \in \mathrm{SL}(2,\mathbb{R})$ are matrices such that $A_1 = A^{-1}A_2A$ then $\mathrm{Trace}(A_1) = \mathrm{Trace}(A^{-1}A_2A) = \mathrm{Trace}(A_2)$. You might first want to show that $\mathrm{Trace}(AB) = \mathrm{Trace}(BA)$ for any two matrices A, B.)

Exercise 10.3
Let $\gamma(z) = z + b$. If $b > 0$ then show that γ is conjugate to $\gamma(z) = z + 1$. If $b < 0$ then show that γ is conjugate to $\gamma(z) = z - 1$. Are $z \mapsto z - 1, z \mapsto z + 1$ conjugate?

Exercise 11.1
Show that two dilations $z \mapsto k_1 z$, $z \mapsto k_2 z$ are conjugate (as Möbius transformations of $\mathbb{H}$) if and only if $k_1 = k_2$ or $k_1 = 1/k_2$.

Exercise 11.2
Let $\gamma \in \mathrm{M\ddot{o}b}(\mathbb{H})$ be a hyperbolic Möbius transformation of $\mathbb{H}$. By the above result, we know that γ is conjugate to a dilation $z \mapsto kz$. Find a relationship between $\tau(\gamma)$ and k.

Exercise 11.3
Let $\gamma \in \mathrm{M\ddot{o}b}(\mathbb{D})$ be a elliptic Möbius transformation of $\mathbb{D}$. By the above result, we know that γ is conjugate to a rotation $z \mapsto e^{i\theta}z$. Find a relationship between $\tau(\gamma)$ and θ.

Exercise 12.1
Show that for each $q \in \mathbb{N}$, Γ_q, as defined above, is indeed a subgroup of $\mathrm{M\ddot{o}b}(\mathbb{H})$.

Exercise 12.2
Fix $k > 0$, $k \neq 1$. Consider the subgroup of $\mathrm{M\ddot{o}b}(\mathbb{H})$ generated by the Möbius transformations of $\mathbb{H}$ given by

$$\gamma_1(z) = z + 1, \ \ \gamma_2(z) = kz.$$

Is this a Fuchsian group? (Hint: consider $\gamma_2^{-n}\gamma_1^m\gamma_2^n(z)$.)

Exercise 13.1
Figures 13.2.1 and 13.2.2 illustrate two tessellations of $\mathbb{H}$. What do these tessellations look like in the Poincaré disc $\mathbb{D}$?

Exercise 14.1♭
(Included for completeness only.) Show that a convex hyperbolic polygon is an open subset of $\mathbb{H}$. To do this, first show that a half-plane is an open set. Then show that the intersection of a finite number of open sets is open.

Exercise 14.2

(i) Write $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, $z_1, z_2 \in \mathbb{H}$. Show that the perpendicular bisector of $[z_1, z_2]$ can also be written as

$$\{z \in \mathbb{H} \mid y_2|z - z_1|^2 = y_1|z - z_2|^2\}.$$

(ii) Hence describe the perpendicular bisector of the arc of geodesic between $1 + 2i$ and $(6 + 8i)/5$.

Exercise 15.1
Let $\Gamma = \{\gamma_n \mid \gamma_n(z) = 2^n z,\ n \in \mathbb{Z}\}$. This is a Fuchsian group. Choose a suitable $p \in \mathbb{H}$ and construct a Dirichlet polygon $D(p)$.

Exercise 16.1
Take $\Gamma = \{\gamma_n \mid \gamma_n(z) = 2^n z, n \in \mathbb{Z}\}$. Calculate the side-pairing transformations for the Dirichlet polygon calculated in Exercise 15.1.

Exercise 17.1♭
Convince yourself that the above two claims [defining elliptic cycles] are true.

Exercise 17.2♭

(i) Show that $\gamma_{v_0,s_0}, \gamma_{v_i,s_i}$ have the same order.

(ii) Show that if γ has order m then so does γ^{-1}.

Exercise 18.1
Check the assertion in example (v) above, i.e. show that if $\Gamma = \langle a, b \mid a^4 = b^2 = (ab)^2 = e\rangle$ then Γ contains exactly 8 elements.

Exercise 19.1♭
Take a hyperbolic quadrilateral such that each pair of opposing sides have the same length. Define two side-pairing transformation γ_1, γ_2 that pair each pair of opposite sides. See Figure 24.2.1. Show that there is only one elliptic cycle and determine the associated elliptic cycle transformation. When do γ_1 and γ_2 generate a Fuchsian group?

Exercise 20.1
Consider the polygon in Figure 24.2.2. The side-pairing transformations are:

$$\gamma_1(z) = z + 2,\ \gamma_2(z) = \frac{z}{2z + 1}.$$

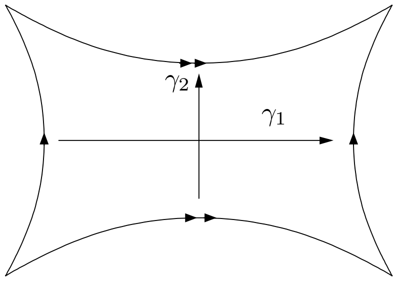

Figure 24.2.1: A hyperbolic quadrilateral with opposite sides paired.

What are the elliptic cycles? What are the parabolic cycles? Use Poincaré's Theorem to show that the Fuchsian group generated by γ_1, γ_2 is discrete and has the polygon in Figure 24.2.2 as a fundamental domain. Use Poincaré's Theorem to show that the group generated by γ_1, γ_2 is the free group on 2 generators.

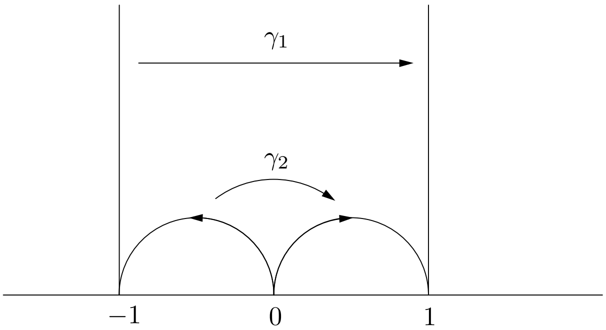

Figure 24.2.2: A fundamental domain for the free group on 2 generators.

Exercise 20.2
Consider the hyperbolic quadrilateral with vertices

$$A = -\left(1 + \frac{\sqrt{2}}{2}\right),\ B = i\frac{\sqrt{2}}{2},\ C = \left(1 + \frac{\sqrt{2}}{2}\right),\ \text{and } \infty$$

and a right-angle at B, as illustrated in Figure 24.2.3.

(i) Verify that the following Möbius transformations are side-pairing transformations:

$$\gamma_1(z) = z + 2 + \sqrt{2}, \quad \gamma_2(z) = \frac{\frac{\sqrt{2}}{2}z - \frac{1}{2}}{z + \frac{\sqrt{2}}{2}}.$$

(ii) By using Poincaré's Theorem, show that these side-pairing transformations generate a Fuchsian group. Give a presentation of Γ in terms of generators and relations.

Exercise 21.1
Consider the hyperbolic polygon illustrated in Figure 24.2.4 with the side-pairing transformations as indicated (note that one side is paired with itself). Assume that $\theta_1+\theta_2+\theta_3 = 2\pi$ (one can show that such a polygon exists).

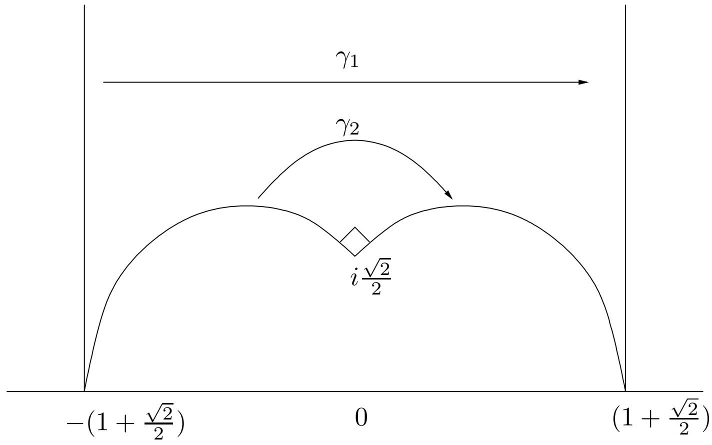

Figure 24.2.3: A hyperbolic quadrilateral.

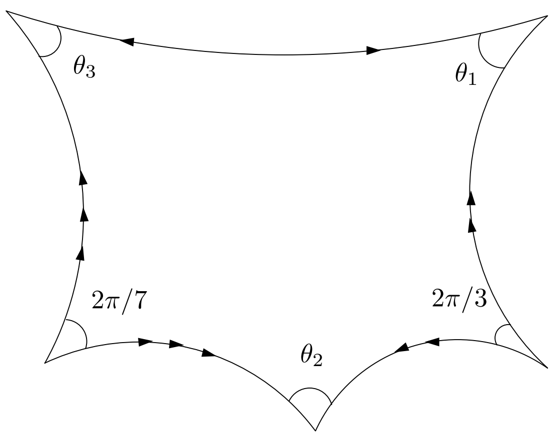

Figure 24.2.4: A hyperbolic polygon with sides paired as indicated.

(i) Show that there are 3 non-accidental cycles and 1 accidental cycle.

(ii) Show that the side-pairing transformations generate a Fuchsian group Γ and give a presentation of Γ in terms of generators and relations.

(iii) Calculate the signature of Γ.

Exercise 21.2
Consider the regular hyperbolic octagon with each internal angle equal to θ and the sides paired as indicated in Figure 24.2.5. Use Exercise 7.3 to show that such an octagon exists provided $\theta \in [0, 3\pi/4)$.

For which values of θ do $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ generate a Fuchsian group Γ_θ? In each case when Γ_θ is a Fuchsian group write down a presentation of Γ_θ, determine the signature $\mathrm{sig}(\Gamma_\theta)$ and briefly describe geometrically the quotient space $\mathbb{H}/\Gamma_\theta$.

Exercise 21.3
This exercise works through the above [in Lecture 21] calculations in the case when we allow parabolic cycles.

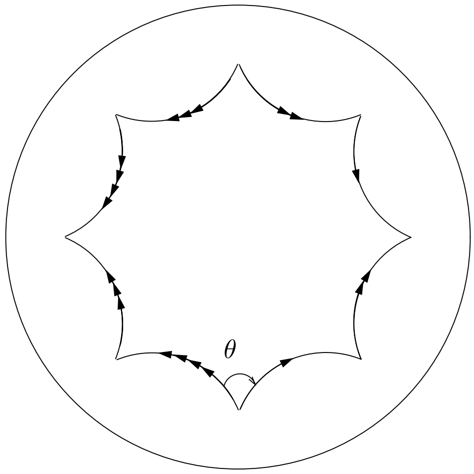

Figure 24.2.5: See Exercise 21.2.

Let Γ be a Fuchsian group and let D be a Dirichlet polygon for D. We allow D to have vertices on $\partial\mathbb{H}$, but we assume that D has no free edges (so that no arcs of $\partial\mathbb{H}$ are edges). We also assume that no side of D is paired with itself.

The space $\mathbb{H}/\Gamma$ then has a genus (heuristically, the number of handles), possibly some marked points, and cusps. The cusps arise from gluing together the vertices on parabolic cycles and identifying the sides on each parabolic cycle.

(i) Convince yourself that the $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$ has genus 0, one marked point of order 3, one marked point of order 2, and one cusp.

(Hint: remember that a side is not allowed to be paired to itself.)

Suppose that $\mathbb{H}/\Gamma$ has genus g, r marked points of order $m_1, \ldots, m_r$, and c cusps. We define the signature of Γ to be

$$\mathrm{sig}(\Gamma) = (g; m_1, \ldots, m_r; c).$$

(ii) Using the Gauss-Bonnet Theorem, show that

$$\mathrm{Area}_{\mathbb{H}}(D) = 2\pi \left((2g-2) + \sum_{j=1}^{r} \left(1 - \frac{1}{m_j} \right) + c \right).$$

(iii) Show that if $c \geq 1$ then

$$\mathrm{Area}_{\mathbb{H}}(D) \geq \frac{\pi}{3}$$

and that this lower bound is achieved for just one Fuchsian group (which one?).