# How to fail Hyperbolic Geometry

## §1 Introduction

These notes describe some of the most common misunderstandings and mistakes that occur almost every year. The section headings contain the most common mistakes students make. If you make these mistakes in the exam then you are throwing marks away—you have been warned!

## §2 The upper half-plane is the right half-plane

No it isn't! In Lecture 2 we defined the upper half-plane $\mathbb{H}$ to be the set $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$. It's the region of the complex plane that lies *above* the real axis. It is *not* the region of the complex plane that lies to the right of the imaginary axis. Draw a picture!

## §3 Every circle in the complex plane has a centre on the real line and every straight line is vertical

This isn't true! We saw in Lecture 3 that an arbitrary straight line or circle in $\mathbb{C}$ has an equation of the form

$$\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0 \tag{3.1}$$

where $\alpha, \gamma \in \mathbb{R}$ and $\beta \in \mathbb{C}$. (Straight lines arise from the case $\alpha = 0$, circles from $\alpha \neq 0$.)

Some (but clearly not all) straight lines are vertical, and some (but clearly not all) circles have real centres. Vertical straight lines and circles with real centres correspond to equations of the form (3.1) where $\beta \in \mathbb{R}$. If $\beta$ is real, then $\bar{\beta} = \beta$. Hence vertical straight lines and circles with real centres have equations of the form

$$\alpha z\bar{z} + \beta z + \beta\bar{z} + \gamma = 0$$

with $\alpha, \beta, \gamma \in \mathbb{R}$.

## §4 Möbius transformations are usually multiplied together

No they aren't! Möbius transformations are *composed*. Let $\gamma_1, \gamma_2$ be Möbius transformations. When we write $\gamma_1\gamma_2$, we always mean the *composition* $\gamma_1 \circ \gamma_2(z)$ (i.e. $\gamma_1(\gamma_2(z))$) and *not* the multiplication of the two complex numbers $\gamma_1(z)\gamma_2(z)$. We saw in Lecture 3 that Möbius transformations form a group under composition.

For example, if $\gamma_1(z) = (2z+1)/(z+1)$ and $\gamma_2(z) = z+3$ then $\gamma_1\gamma_2$ denotes the transformation

$$\begin{aligned}\gamma_1\gamma_2(z) &= \gamma_1(\gamma_2(z)) \\ &= \gamma_1(z+3) \\ &= \frac{2(z+3)+1}{(z+3)+1} \\ &= \frac{2z+7}{z+4}\end{aligned}$$

It does not denote the multiplication

$$\gamma_1(z)\gamma_2(z) = \left(\frac{2z+1}{z+1}\right) \times (z+3) = \frac{2z^2+7z+3}{z+1},$$

which isn't even a Möbius transformation. In short: if you get a $z^2$ somewhere then you've multiplied and not composed.

Note also that composition is not necessarily commutative. In the above example, we saw that

$$\gamma_1\gamma_2(z) = \frac{2z+7}{z+4}.$$

However,

$$\gamma_2\gamma_1(z) = \gamma_2\left(\frac{2z+1}{z+1}\right) = \left(\frac{2z+1}{z+1}\right) + 3 = \frac{5z+4}{z+1}.$$

## §5 There's no need to check that a Möbius transformation is a Möbius transformation

Yes there is! A Möbius transformation is a map of the form $(az+b)/(cz+d)$ where $ad-bc>0$. You do need to check that $ad-bc>0$.

There are two places where this is particularly important:

(i) When checking that Möbius transformations form a group, you need to check that the inverse of a Möbius transformation is a Möbius transformation, and the composition of two Möbius transformations is a Möbius transformation.

(ii) When moving an arbitrary geodesic to the imaginary axis, you need to check that the transformation used is a Möbius transformation. (See Lemma 4.3 or Lemma 5.1 in the notes.)

With regard to (ii), suppose that we want to move the geodesic with end-points $\alpha < \beta \in \partial\mathbb{H}$ to the imaginary axis. The end-points of the imaginary axis are $0, \infty$, so we want to move $\alpha$ to either 0 or $\infty$ and $\beta$ to either $\infty$ or 0. That is, we need a transformation of the following form:

$$\frac{z-\alpha}{z-\beta}, \text{ or } \frac{z-\beta}{z-\alpha}.$$

Note that if we calculate '$ad - bc$' for these two transformations we obtain $-\beta + \alpha$ for the first transformation and $-\alpha + \beta$ for the second. As $\alpha < \beta$, we have $-\alpha + \beta > 0$ and so only the second transformation is a Möbius transformation.

## §6 There is no point in normalising Möbius transformations

Yes there is! Recall from Lecture 9 that a Möbius transformation $\gamma(z) = (az+b)/(cz+d)$ is *normalised* if $ad - bc = 1$. We can always normalise a Möbius transformation by dividing the coefficients by $\sqrt{ad-bc}$.

The trace $\tau(\gamma)$ of a Möbius transformation $\gamma$ is defined to be $(a+d)^2$ where $\gamma(z) = (az+b)/(cz+d)$ is in normalised form. We saw that $\gamma$ is elliptic, parabolic or hyperbolic according to whether $\tau(\gamma) \in [0,4), \tau(\gamma) = 4$ or $\tau(\gamma) > 4$, respectively. If we don't normalise, then this classification doesn't work.

For example, consider $\gamma(z) = (z-1)/(4z+5)$. Here $ad - bc = 1 \times 5 - (-1) \times 4 = 9$, so this isn't normalised. Dividing by $\sqrt{9} = 3$ we can write $\gamma(z)$ in normalised form as

$$\gamma(z) = \frac{\frac{1}{3}z - \frac{1}{3}}{\frac{4}{3}z + \frac{5}{3}}.$$

(Note that this is now normalised: $ad - bc = (1/3)(5/3) - (-1/3)(4/3) = 9/9 = 1$.) This has trace

$$\tau(\gamma) = \left(\frac{1}{3} + \frac{5}{3}\right)^2 = 4$$

and so is parabolic. (This can be checked directly by showing that $\gamma$ has a unique fixed point at $z = -1/2 \in \partial\mathbb{H}$.) However, if we hadn't normalised $\gamma$ then working out $(a+d)^2$ would give us $(1+5)^2 = 36$ and we would have incorrectly concluded that $\gamma$ was hyperbolic.

## §7 Hyperbolic, parabolic and elliptic Möbius transformations are easily confused

No they aren't: learn the following table.

| | No. of fixed points in $\mathbb{H}/\mathbb{D}$ | No. of fixed points in $\partial\mathbb{H}/\partial\mathbb{D}$ | Trace | Conjugate to |
|---|---|---|---|---|
| hyperbolic | 0 | 2 | $\tau(\gamma) > 4$ | a dilation $z \mapsto kz,\ k \neq 1$ |
| parabolic | 0 | 1 | $\tau(\gamma) = 4$ | the translation $z \mapsto z + 1$ |
| elliptic | 1 | 0 | $\tau(\gamma) \in [0, 4)$ | a rotation |

Two MSc students once suggested the following mnemonic:

- The number of fixed points in $\mathbb{H}/\mathbb{D}$ is the number of 't's in the words hyperbolic, parabolic, elliptic, respectively.
- The number of fixed points in $\partial\mathbb{H}/\partial\mathbb{D}$ is the number of 'y's plus the number of 'bolic's in the words hyperbolic, parabolic, elliptic, respectively.
- The trace corresponds the number of letters—either more than, or equal, to 4—before 'bolic' in hyperbolic, parabolic (with elliptic transformations corresponding to the remaining possible values of the trace).

(It's probably easier to just learn the table above...)

## §8 The terms fundamental domain, Dirichlet region and Dirichlet polygon are interchangeable

No they aren't! Let $\Gamma$ be a Fuchsian group. A fundamental domain $F$ for $\Gamma$ is (loosely speaking) any open subset of $\mathbb{H}$ whose images under $\Gamma$ tile $\mathbb{H}$. (In fact, in Lecture 13 we said that an open subset $F$ is a fundamental domain for $\Gamma$ if

(i) $\gamma_1(F) \cap \gamma_2(F) = \emptyset$ for $\gamma_1, \gamma_2 \in \Gamma, \gamma_1 \neq \gamma_2$,

(ii) $\bigcup_{\gamma\in\Gamma} \gamma(\mathrm{cl}\, F) = \mathbb{H}$.)

A given Fuchsian group $\Gamma$ may have lots of fundamental domains (see Lecture 13 for some examples).

Let $\Gamma$ be a Fuchsian group. It is not, in general, clear how to write down a fundamental domain for $\Gamma$. An algorithm that will generate a fundamental domain is given in Lecture 14, and a fundamental domain generated in this way is called a *Dirichlet region*, usually denoted in the course by $D(p)$.

$D(p)$ is defined to be

$$D(p) = \bigcap_{\gamma\in\Gamma\setminus\{\mathrm{id}\}} H_\gamma(p)$$

i.e. as an intersection of possibly infinitely many half-planes. In the vast majority (but not all—although it is beyond the scope of the course to give an example) of cases, this intersection is actually the intersection of just finitely many half-planes. (See the worked examples in Lecture 15.) In this case, $D(p)$ is a hyperbolic polygon and we call it a *Dirichlet polygon*.

## §9 "If and only if" statements only need proving in one direction

No, they need proving in both directions. For example, recall that a Möbius transformation is hyperbolic if it has two fixed points on the boundary. If you're asked to show that a Möbius transformation $\gamma$ is hyperbolic if, and only if, it is conjugate to a dilation then this means you have to show two things: (i) if $\gamma$ is hyperbolic then it is conjugate to a dilation, and (ii) if $\gamma$ is conjugate to a dilation then it is hyperbolic. It's not enough just to do one implication!

## §10 Poincaré's Theorem is impossible to remember

No it isn't, although I admit that the statement of Poincaré's Theorem is rather long and complicated. The statement breaks down into 3 section: (i) defining notation, (ii) the hypotheses, (iii) the conclusions. Also, remember to check whether you're working in the case of no boundary vertices (Lecture 19), or boundary vertices but no free edges (Lecture 20).

The part that most people forget is the hypothesis that no side of the hyperbolic polygon is paired with itself. This is important because if this assumption is omitted then it's possible to miss out one of the relations in the group. For example, in §20.3 when we illustrate Poincaré's Theorem in the case of the modular group, if we hadn't introduced the extra vertex at $C = i$ then we wouldn't have obtained the relation $b^2 = e$. The remaining hypotheses are that the Elliptic Cycle Condition holds and, in the case of boundary vertices (but no free edges), the Parabolic Cycle Condition holds.

There are essentially three points to remember in the conclusions: (i) the side-pairing transformations generate a Fuchsian group, (ii) the polygon is a fundamental domain, and (iii) the elliptic cycles can be used to give a presentation of the group in terms of generators and relations. (You would need to elaborate on the final point to score full marks in the exam by writing down how to obtain such a presentation.)

## §11 It's possible to do well in the hyperbolic geometry exam without going to the lectures and without doing any of the exercises

Sadly not ;-)