Feedback on: Hyperbolic Geometry, Jan 2018

Section A

A1 (i) This is bookwork (see §3.2 in the lecture notes). Some people did the calculation in the opposite order to how I did it, starting with $\beta z + \bar{\beta}\bar{z} + \gamma = 0$, substituting $z = x + iy$, and then arriving at $ax + by + c = 0$; this is absolutely fine.

(ii) Horizontal lines have constant y co-ordinate, so correspond to equations of the form $ax + by + c = 0$ with $a = 0$. In the calculation you did in (i) you see that $\beta = (a - ib)/2$. Hence the line is horizontal iff $a = 0$ iff β is purely imaginary.

(iii) The two points $-1 + 2i, 3 + 2i$ clearly lie on a horizontal line. There are several ways of calculating the equation of this line (some of which are slightly quicker) than the one given below; any argument that gave the right answer is fine.

From (ii) we know that β is purely imaginary, so set $\beta = ib$ for some $b \in \mathbb{R}$. Hence if $-1 + 2i, 3 + 2i$ lie on $\beta z + \bar{\beta}\bar{z} + \gamma = 0$ then we get the simultaneous equations

$$ib(-1 + 2i) - ib(-1 - 2i) + \gamma = 0, \quad ib(3 + 2i) - ib(3 - 2i) + \gamma = 0.$$

Simplifying these we get $-4b + \gamma = 0$. Take $b = 1, \gamma = 4$. Then we have equation $iz - i\bar{z} + 4 = 0$. (Aside: any scalar multiple of this equation is also correct.)

A2 (i) This is a standard definition: $\gamma_1, \gamma_2 \in \text{Möb}(\mathbb{H})$ are conjugate if there exists $g \in \text{Möb}(\mathbb{H})$ such that $\gamma_1 = g^{-1}\gamma_2 g$.

(ii) Simply note that $\gamma(z_0) = z_0$ iff $kz_0 = z_0$ iff $z_0 = 0, \infty$. Hence γ has two fixed points on the boundary and none in $\mathbb{H}$, so is hyperbolic.

(iii) This is a proof from the course (it's (i) $\Rightarrow$ (iii) in Proposition 11.2.1). The question gives you several hints on the steps you need to take.

A3 (i) This is a standard definition. An open subset $F \subset \mathbb{D}$ is a fundamental domain for Γ if

(i) $\bigcup_{\gamma \in \Gamma} \gamma(\text{cl}\, F) = \mathbb{D}$,

(ii) $\gamma_1(F) \cap \gamma_2(F) = \emptyset$ if $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$.

(A common mistake was to write $\bigcup_{\gamma \in \Gamma \setminus \{\text{id}\}}$ in (i).)

(ii) The tessellation looks like a pizza divided into 5 equal pieces. (As you know, drawing pictures in LaTeX is a pain so I won't include the picture.)

A4 (i) This is a standard definition. Γ is a Fuchsian group if it is a discrete subgroup of $\text{Möb}(\mathbb{H})$.

The orbit of $z \in \mathbb{H}$ is defined to be $\Gamma(z) = \{\gamma(z) \mid \gamma \in \Gamma\}$. (It's the set of every point in $\mathbb{H}$ you can get to by applying elements of Γ to the point z; in particular, it's a subSET of $\mathbb{H}$, not a subGROUP of Γ.)

(ii) Here $\Gamma(1 + i) = \{2^n + 2^n i \mid n \in \mathbb{Z}\}$. These points lie along the diagonal $y = x$ in the Argand diagram (many of you drew them as if they were on the line $y = x^2$ or similar).

(iii) There's a few ways of doing this. Here's one: take $z = 0 \in \partial\mathbb{H}$. Then

$$\begin{aligned}
\Gamma(0) &= \left\{ \gamma(0) \mid \gamma(z) = \frac{az + b}{cz + d},\ a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\} \\
&= \left\{ \frac{b}{d} \mid \exists a, c \in \mathbb{Z} \text{ s.t. } ad - bc = 1 \right\} \\
&= \mathbb{Q} \cup \{\infty\}.
\end{aligned}$$

This is not a discrete subset of $\partial\mathbb{H}$ as any neigbourhood of 0 contains a non-zero rational.

Some of you had a more ingeneous argument. Note that both $z \mapsto z+n, z \mapsto -1/z \in \mathrm{PSL}(2,\mathbb{Z})$. Hence the composition $z \mapsto -1/(z+n) \in \mathrm{PSL}(2,\mathbb{Z})$. Take $z=0$. Then $0 \in \Gamma(0)$, for example because $z \mapsto z/(z+1) \in \mathrm{PSL}(2,\mathbb{Z})$ and maps 0 to 0. Also $z \mapsto -1/(z+n)$ maps 0 to $-1/n$. Hence $\Gamma(0) \supset \{0, -1/n \mid n \in \mathbb{N}\}$, and so is not discrete.

Some people tried to argue that the orbit of some point is not discrete by looking at its images under all translations $z \mapsto z+b, b \in \mathbb{R}$. This doesn't work, as $z \mapsto z+b \notin \mathrm{PSL}(2,\mathbb{Z})$ unless $b \in \mathbb{Z}$.

Section B

B5 (i) This is exercise 2.3; we spoke about this at length in one of the support classes.

(ii) Let $\sigma : [a,b] \to X$ be a path. Then

$$\mathrm{length}_\rho(\sigma) = \int_a^b \rho(\sigma(t))|\sigma'(t)|\,dt$$

and

$$\begin{aligned}\mathrm{length}_\rho(\gamma\circ\sigma) &= \int_a^b \rho(\gamma\circ\sigma(t))|(\gamma\circ\sigma)'(t)|\,dt \\ &= \int_a^b \rho(\gamma(\sigma(t)))|\gamma'(\sigma(t))|\,|\sigma'(t)|\,dt\end{aligned}$$

(where we have used the chain rule). Hence $\mathrm{length}_\rho(\sigma) = \mathrm{length}_\rho(\gamma\circ\sigma)$ iff

$$\int_a^b \big(\rho(\gamma(\sigma(t)))|\gamma'(\sigma(t))| - \rho(\sigma(t))\big)\,|\sigma'(t)|\,dt$$

iff $\int_\sigma (\rho\circ\sigma|\gamma'| - \rho) = 0$. By the hint in the question, this happens iff $\rho(\gamma(z))|\gamma'(z)| = \rho(z)$ for all $z \in X$.

We claim that showing that $\rho(\gamma(z))|\gamma'(z)| = \rho(z)$ for all $z \in X$ implies that γ is an isometry. Note that σ is a path from z_1 to z_2 iff $\gamma\sigma$ is a path from $\gamma(z_1)$ to $\gamma(z_2)$. Hence taking the infimum over all paths σ form z_1 to z_2 in the equation labelled (*) in the question proves this claim.

(iii) All I wanted you to do here was to use the two facts about $\mathrm{Im}\,\gamma(z)$ and $|\gamma'(z)|$ (when γ is a Möbius transformation) given in the question to check that, when $\rho(z) = 1/\mathrm{Im}(z)$, the equation $\rho(\gamma(z))|\gamma'(z)| = \rho(z)$ holds.

This follows as in this case

$$\rho(\gamma(z))|\gamma'(z)| = \frac{|cz+d|^2}{ad-bc} \times \frac{1}{\mathrm{Im}(z)} \times \frac{ad-bc}{|cz+d|^2} = \frac{1}{\mathrm{Im}(z)} = \rho(z).$$

(iv) This is exercise 5.5 parts (ii), (iii) in the notes.

(v) Very few of you tried this. It's actually very straightforward (and very similar to exercise 5.5(ii)). First note that the Euclidean metric corresponds to taking $\rho(z) = 1$. Take any $a+ib \in \mathbb{C}$. Note that $|T'_{a,b}(x,y)| = 1$. Hence if $\rho(T_{a,b}(x+iy))|T'_{a,b}(x+iy)| = \rho(x+iy)$ then $\rho(x+a+i(y+b)) = \rho(x+iy)$. Now take $(x,y) = (0,0)$. Then $\rho(a+ib) = \rho(0)$ for all $a+ib \in \mathbb{C}$. Hence ρ is a constant and so the metric is a constant multiple of the Euclidean metric.

B6 Almost all of this question focusses on what we covered on the Gauss-Bonnet Theorem in Lecture 7.

(i) This is the proof of the Gauss-Bonnet Theorem (for a triangle); see Theorem 7.2.1 in the notes. Common mistakes included: (i) Not pointing out that applying a sequence of Möbius transformations so that the triangle has one side that lies along the unit circle in $\mathbb{C}$ does not change the area (as Möbius transformations are area-preserving) nor angles (as Möbius transformations are conformal); (ii) assuming that one of the angles is a right-angle; (iii) only proving the theorem in the case where one vertex is at ∞.

(ii) Split the quadrilateral into two triangles by drawing in the diagonal between (say) vertices B and D. This splits the internal angle β at B into β_1 and β_2 with $\beta = \beta_1 + \beta_2$. Ditto for δ. Many of you assumed that $\beta_1 = \beta_2 = \beta/2$ (and similarly for δ); there is no reason why drawing in this diagonal will bisect the angles (I did talk about that in a support class). The result then follows by adding together the areas of the two traingles formed and using the Gauss-Bonnet Theorem.

(iii) The area of a circle is calculated in Exercise 6.5 in the notes. A common mistake occured when calculating $\int_{\rho=0}^{r} \frac{\rho}{(1-\rho^2)^2}\, d\rho$ and working out

$$\left.\frac{1}{1-\rho^2}\right|_0^r.$$

Note that when you put in the lower limit 0, you don't get 0. Indeed,

$$\left.\frac{1}{1-\rho^2}\right|_0^r = \frac{1}{1-r^2} - 1 = \frac{r^2}{1-r^2}.$$

(iv) Let $Q(r)$ be the hyperbolic quadrilateral with vertices at $\pm r, \pm ir$. (Draw it!)

First note that this quadrilateral is invariant under rotation through 90, 180, and 270 degrees. Also recall that rotations around the origin are Möbius transformations. This shows that each side of $Q(r)$ has the same length and that all the internal angles are equal. Let $\alpha(r)$ be the common value of each internal angle of $Q(r)$.

We want to investigate what happens to $\alpha(r)$ as $r \to 0$ (i.e. the quadrilateral becomes 'degenerate' in a sense) and when $r \to 1$ (i.e. the vertices approach the boundary).

By part (ii) of this question, we know that $\mathrm{Area}_{\mathbb{D}}\, Q(r) = 2\pi - 4\alpha(r)$.

Clearly (draw a picture!) we have that $Q(r) \subset C_r$. Hence $\mathrm{Area}_{\mathbb{D}}\, Q(r) \leq \mathrm{Area}_{\mathbb{D}}\, C_r = 4\pi r^2/(1-r^2)$. Note that $4\pi r^2/(1-r^2) \to 0$ as $r \to 0$. Hence $\lim_{r\to 0} \mathrm{Area}_{\mathbb{D}}\, Q(r) = 0$. Hence $\lim_{r\to 0} \alpha(r) = \pi/2$.

As $r \to 1$, the vertices converge to points on the boundary. The internal angle at a vertex on the boundary is 0. Hence $\lim_{r\to 1} \alpha(r) = 0$.

By the Intermediate Value Theorem (and noting that $\alpha(r)$ is continuous in r), it follows that for any $\alpha \in [0, \pi/2)$ (note which bracket is open and which is closed) there exists $r \in (0, 1]$ such that $\alpha(r) = \alpha$.

Conversely, if $\alpha > \pi/2$ then, by (ii) in this question, $Q(r)$ has negative area; a contradiction.

(v) If k quadrilaterals meet at each vertex then the internal angle must be $\alpha = 2\pi/k$. By part (ii) of this question, we have $2\pi - 4\alpha > 0$. This rearranges to $k > 4$.

B7 (i) Note that $\gamma(z_0) = z_0$ iff $(az_0 + b)/(cz_0 + d) = z_0$ iff $cz_0^2 + (d-a)z_0 - b = 0$. It follows from the quadratic formula that there is a unique (real) fixed point for γ iff we have $(d-a)^2 + 4bc = 0$.

Aside: The point of this is that it gives a quick, checkable, condition to see if a Möbius transformation is parabolic or not *without* having to calculate $\tau(\gamma)$ (more on which below).

(ii) As geodesics are uniquely determined by their endpoints, all you need do here is to check that γ_1 maps 0 to 0 and -1 to 1.

(iii) (Apologies for the typo in the question.)

An elliptic cycle $\mathcal{E}$ satisfies the ECC if there exists $m \in \mathbb{N}$ such that $m\,\mathrm{sum}(\mathcal{E}) = 2\pi$.

A parabolic cycle $\mathcal{P}$ satisfies the PCC if the associated parabolic cycle transformation is either parabolic or the identity. (Several of you missed out the identity.)

(iv) Almost everybody correctly worked out the elliptic and parabolic cycles and their transformations. For the record, they are:

* Parabolic cycle $\mathcal{P}_1 = 0$, parabolic cycle transformation γ_1,
* Parabolic cycle $\mathcal{P}_2 = -1 \mapsto 1$, parabolic cycle transformation $\gamma_1^{-1}\gamma_2$,
* Elliptic cycle $\mathcal{E} = i\ell$, $\mathrm{sum}(\mathcal{E}) = 2\theta$, elliptic cycle transformation γ_2.

Now we need to check that the ECC and PCC holds for the above.

Consider $\mathcal{P}_1$. Note that the formula given for γ_1 in the question **IS NOT NORMALISED SO YOU CANNOT JUST WRITE DOWN '$(a+d)^2 = 4$' AND HOPE FOR THE BEST** (although, by chance, this does give the right answer). Instead, part (i) of the question gives an easy-to-check condition to see if a Möbius transformation is parabolic or not. From (i), we see that γ_1 is parabolic iff

$$(1-k)^2 + 4 \times 0 = (1-k)^2 = 0$$

i.e. if $k = 1$.

Now consider $\mathcal{P}_2$ with $k = 1$. Then $\gamma_1^{-1}\gamma_2$ has matrix

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1-\ell^2 & -2\ell^2 \\ 2 & 1-\ell^2 \end{pmatrix} = \begin{pmatrix} 1-\ell^2 & -2\ell^2 \\ 2\ell^2 & 3\ell^2+1 \end{pmatrix}.$$

(Aside: this is not normalised so again one cannot just do '$ad - bc = 4$'.) We can use (i) again to see if this is parabolic. Note that

$$(3\ell^2 + 1 - 1 + \ell^2)^2 + 4(-2\ell^2)(2\ell^2) \equiv 0.$$

Hence $\mathcal{P}_2$ satisfies the PCC for all values of ℓ. (Very few of you checked to see if $\mathcal{P}_2$ satisfied the PCC.)

Consider $\mathcal{E}_2$. This satisfies the ECC if there exists an integer $m \geq 1$ such that $\theta = \pi/m$. When γ_1, γ_2 generate a Fuchsian group (i.e. when $k = 1, \theta = \pi/m$), Poincaré's Theorem tells us that it has presentation

$$\Gamma = \langle a, b \mid b^m = e \rangle.$$

(v) (I won't redraw the picture here—see the exam paper.)

Basic geometry says that the internal angle at x is θ. Hence $\sin\theta = \ell/r$.

The (Euclidean) right-angled triangle with vertices at x, 0 and $i\ell$ has sides of length $r-1, \ell$ and r. By Pythagoras, we have $(r-1)^2 + \ell^2 = r^2$, hence (multiplying out and simplifying) $r = (\ell^2+1)/2$. (Note that Pythagoras' Theorem does *not* say that $(r-1)^2+(i\ell)^2 = (r-1)^2-\ell^2 = r^2$—it's the *length* of the sides that appear in Pythagoras' Theorem, not the co-ordinates of the vertices.)

Hence $\sin\theta = 2\ell/(\ell^2+1)$.

Many of you stopped here. The question wants you to find an explicit value of ℓ which will give a Fuchsian group with presentation $\langle a, b \mid b^6 = e\rangle$. This means that we need to find ℓ for which $\theta = \pi/6$. The question reminds you that $\sin\pi/6 = 1/2$, so we have to find ℓ such that

$$\frac{1}{2} = \frac{2\ell}{\ell^2+1}.$$

This gives the quadratic equation $\ell^2 - 4\ell + 1 = 0$ which has solutions $\ell = 2 \pm \sqrt{3}$. As we want $\ell > 1$, we must have that $\ell = 2+\sqrt{3}$.

Section C Many of you taking the Level 4/6 version of the course (particularly those taking the Level 4 version) significantly underperformed in Section C compared to Sections A, B. I did warn you (and it's clear from previous exams) that Section C is worth 1/3rd of the course.

C8 (i) This is a standard definition from the course: Γ acts properly discontinuously on X if, for all compact subsets $K \subset X$ and for all $x \in X$, we have that $\operatorname{card}\{\gamma \in \Gamma \mid \gamma(x) \in K\} < \infty$.

(ii) (a) This does act properly discontinuously. Let $K \subset \mathbb{C}$ be compact. Then K is closed and bounded, so $K \subset [-M, M] \times [-M, M]$ for some $M \in \mathbb{N}$. Let $x+iy \in \mathbb{C}$. Then there are at most $(2M+1)^2$ points of the form $(x+p)+i(y+q) \in [-M, M]\times[-M, M]$ as p, q vary over $\mathbb{Z}$. Hence $\operatorname{card}\{p, q \in \mathbb{Z} \mid \gamma_{p,q}(x+iy) \in K\} \leq (2M+1)^2$.

(b) This does not act properly discontinuously. Take $K = \{z \in \mathbb{C} \mid |z| = 1\}$. Then K is compact. Let $z = 1$. Then $\gamma_\theta(z) \in K$ for all θ. Note that there are infinitely many such θ.

(Alternatively, you could use the criterion that Γ acts properly discontinuously iff every orbit is discrete and every stabiliser is finite.)

(iii) This is Exercise 23.8 in the notes.

C9 (i) This is a standard definition: $\xi \in \Lambda(\Gamma)$ if for some (hence any) $z \in \mathbb{D}$ there exists $\gamma_n \in \Gamma$ such that $|\gamma_n(z) - \xi| \to 0$ as $n \to \infty$.

(ii) This is asking for the proof of Proposition 26.1.1 in the notes.

(iii) This is Proposition 24.4.5(ii) in the notes.

(iv) A result from the course showed that in $\operatorname{card}\Lambda(\Gamma) \geq 3$ then $\operatorname{card}\Lambda(\Gamma) = \infty$. By (iii) we want to look for fixed points of hyperbolic transformations in Γ; these fixed points will be in $\Lambda(\Gamma)$.

Note that $\gamma_1\gamma_2$ and $\gamma_1^2\gamma_2 \in \Gamma$. It's straightforward to check that $\gamma_1\gamma_2(z) = (5z+2)/(2z+1)$ and has fixed points at $3 \pm 2\sqrt{2}$. Also $\gamma_1^2\gamma_2(z) = (9z+4)/(2z+1)$ and has fixed points at $5 \pm 2\sqrt{6}$. (There are many other ways of finding points in $\Lambda(\Gamma)$.)

Hence $\Lambda(\Gamma) \supset \{3 \pm 2\sqrt{2}, 5 \pm 2\sqrt{6}\}$. Hence $\Lambda(\Gamma)$ is infinite.

Charles Walkden
21st January 2018