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Recap

- An isolated singularity at z_0 of f is a pole of order m if f has Laurent series

$$f(z) = \frac{b_m}{(z-z_0)^m} + \dots + \frac{b_1}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad b_m \neq 0$$

valid on $0 < |z-z_0| < R$ (for some $R > 0$)

- If $\frac{f - f(z)}{g(z)} = \frac{p(z)}{q(z)}$ (p, q holomorphic on an appropriate domain)

- $p(z_0) \neq 0$

- q has a zero of order m at z_0

then f has a pole of order m at z_0

Residues & Cauchy's Residue Thm

(2)

Suppose f is holomorphic on D except for an isolated singularity at z_0 . Expand f as a Laurent series on $0 < |z - z_0| < R$ (for some $R > 0$)

$$f(z) = \dots + \frac{b_m}{(z - z_0)^m} + \dots + \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + \dots + a_n(z - z_0)^n + \dots$$

Defn The residue of f at z_0 is $\text{Res}(f, z_0) := b_1$
= coefficient of the term $(z - z_0)^{-1}$.

Suppose f has a pole of order m at z_0 . Then it has a Laurent series

$$f(z) = \frac{b_m}{(z - z_0)^m} + \dots + \frac{b_1}{z - z_0} + \sum_{n=0}^{\infty} a_n(z - z_0)^n \quad b_m \neq 0$$

order of the pole. residue at the pole.

valid on $0 < |z - z_0| < R$.

Calculating residues

Let's first calculate the residue at a simple pole.

Lemma ① If f has a simple pole at z_0 then

$$\text{Res}(f, z_0) = \lim_{z \rightarrow z_0} (z - z_0) f(z)$$

$$\text{② If } f(z) = \frac{p(z)}{q(z)} \quad p, q \text{ diff'ble, } p(z_0) \neq 0 \\ q(z_0) = 0 \quad q'(z_0) \neq 0$$

$$\text{then } \text{Res}(f, z_0) = \frac{p(z_0)}{q'(z_0)}$$

Pf ① Suppose f has a simple pole at z_0 . Then f has ③ a Laurent series:

$$f(z) = \frac{b_1}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{on } 0 < |z-z_0| < R, \quad b_1 \neq 0.$$

So $(z-z_0)f(z) = b_1 + a_0(z-z_0) + a_1(z-z_0)^2 + \dots + a_n(z-z_0)^{n+1}$
 $\rightarrow b_1 = \operatorname{Res}(f, z_0) \quad \text{on } z \rightarrow z_0.$

② First note the hypotheses on p, q imply that f has a simple pole at z_0 . Hence by ①

$$\begin{aligned} \operatorname{Res}(f, z_0) &= \lim_{z \rightarrow z_0} (z-z_0)f(z) = \lim_{z \rightarrow z_0} (z-z_0) \frac{p(z)}{q(z)} \\ &= \lim_{z \rightarrow z_0} \frac{p(z)}{\left(\frac{q(z)-q(z_0)}{z-z_0} \right)} = \frac{p(z_0)}{q'(z_0)} \quad \text{on } q'(z_0) \neq 0. \end{aligned}$$

□.

Example $f(z) = \frac{\cos \pi z}{1-z^5} = \frac{p(z)}{q(z)}$. Calculate $\operatorname{Res}(f, 1)$.

$q(z) = 1-z^5$ has a simple zero at $z=1$ (either by noting that $1-z^5 = (1-z)(1+z+z^2+z^3+z^4)$ or by noting that $q(1)=0, q'(1)=-5z^4, q'(1) \neq 0$).

So by the above lemma

$$\operatorname{Res}(f, 1) = \left. \frac{\cos \pi z}{-5z^4} \right|_{z=1} = \frac{-1}{-5} = \frac{1}{5}.$$

How to calculate the residue at a pole of order m : (4)

Lemma Suppose f has a pole of order m at z_0 . Then

$$\text{Res}(f, z_0) = \lim_{z \rightarrow z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right].$$

Pf We know f has a Laurent series on $0 < |z - z_0| < R$

$$\frac{b_m}{(z - z_0)^m} + \dots + \frac{b_1}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad b_m \neq 0.$$

$$\Rightarrow (z - z_0)^m f(z) = b_m + (z - z_0) b_{m-1} + \dots + (z - z_0)^{m-1} b_1 \\ + \sum_{n=0}^{\infty} a_n (z - z_0)^{n+m}$$

$$\Rightarrow \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right] = (m-1)! b_1 + \sum_{n=0}^{\infty} a_n (n+m)(n+m-1) \dots \\ \dots (n+2) (z - z_0)^{n+1}$$

$$\rightarrow (m-1)! b_1 = (m-1)! \text{Res}(f, z_0)$$

or $z \rightarrow z_0$

□

Example Let $f(z) = \left(\frac{z+1}{z-1}\right)^3$. Calculate $\text{Res}(f, 1)$.

By the above lemma (noting that $z=1$ is a pole of order 3)

$$\text{Res}(f, 1) = \lim_{z \rightarrow 1} \frac{1}{2!} \frac{d^2}{dz^2} \left[(z-1)^3 f(z) \right]$$

$$= \lim_{z \rightarrow 1} \frac{1}{2!} \frac{d^2}{dz^2} \left[(z+1)^3 \right]$$

$$= \lim_{z \rightarrow 1} \frac{1}{2!} 3 \cdot 2 \cdot (z+1) = 6.$$

Alternatively, we could calculate the Laurent series ⑤
at $z = 1$.

Let $w = z - 1$. So $z = w + 1$. So

$$f(z) = \left(\frac{z+1}{z-1}\right)^3 = \left(\frac{w+2}{w}\right)^3 = \frac{w^3 + 6w^2 + 12w + 8}{w^3}$$

$$= \frac{8}{w^3} + \frac{12}{w^2} + \frac{6}{w} + 1.$$

$$= \frac{8}{(z-1)^3} + \frac{12}{(z-1)^2} + \frac{6}{z-1} + 1. \quad \text{valid if } z \neq 1.$$

pole of order 3.

at $z = 1$

$$\text{Res}(f, 1) = 6.$$

①

Recap

Suppose f has an isolated singularity at z_0 . Write f as a Laurent series valid on $0 < |z - z_0| < R$

$$f(z) = \dots + \frac{b_n}{(z-z_0)^n} + \dots + \frac{b_2}{(z-z_0)^2} + \frac{\textcircled{b}_1}{z-z_0} + \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

b_1 = residue of f at z_0
 $= \operatorname{Res}(f, z_0)$.

Example Calculate the singularities & their residues (2)

of $f(z) = \frac{1}{z^2 \sin z}$.

This has singularities at $z=0$, $z=k\pi$ ($k \in \mathbb{Z}, k \neq 0$).

Calculate the residue at 0:

$$\frac{1}{z^2 \sin z} = \frac{1}{z^2 \left(z - \frac{z^3}{3!} + \text{h.o.t.} \right)} = \frac{1}{z^3 \left(1 - \frac{z^2}{3!} + \text{h.o.t.} \right)}$$

(h.o.t. = higher order terms)

$$= \frac{1}{z^3} \times \left(1 - \frac{z^2}{3!} + \text{h.o.t.} \right)^{-1} \quad \left[(1-x)^{-1} \approx 1+x \text{ if } x \text{ is small.} \right]$$

$$= \frac{1}{z^3} \times \left(1 + \frac{z^2}{3!} + \text{h.o.t.} \right)$$

$$= \frac{1}{z^3} + \left(\frac{1}{3!} z \right) + \text{h.o.t.}$$

pole of order 3. $\text{Res}(f, 0) = \frac{1}{3!} = \frac{1}{6}$.

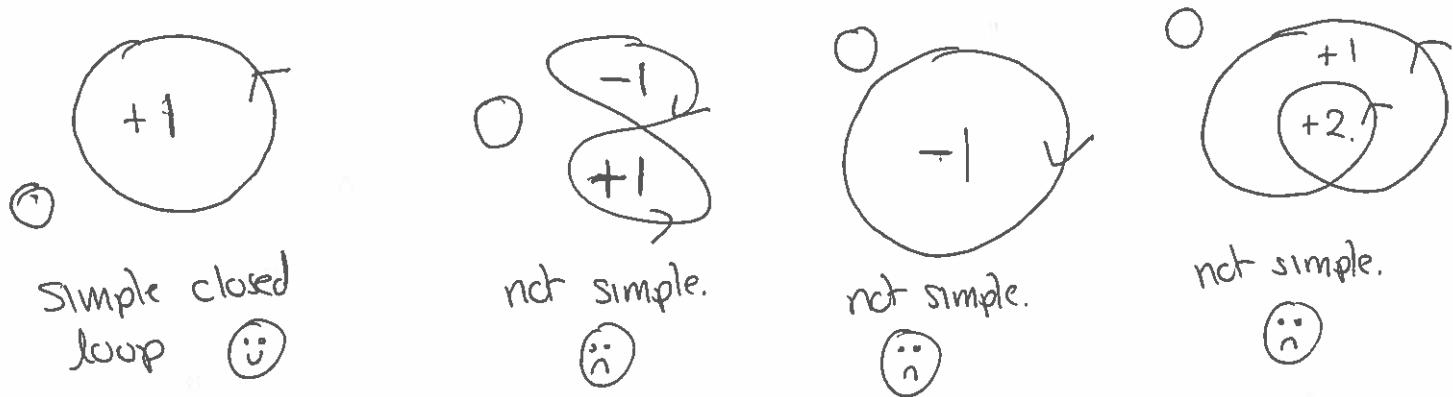
For $k\pi$, $k \neq 0$: Write $f(z) = \frac{p(z)}{q(z)}$ $p(z) = 1$
 $q(z) = z^2 \sin z$.

Then $p(k\pi) \neq 0$, $q'(k\pi) = 0$, $q''(k\pi) = (-1)^k (k\pi)^2 \neq 0$

Hence $\text{Res}(f, k\pi) = \frac{p(k\pi)}{q''(k\pi)} = \frac{(-1)^k}{(k\pi)^2}$.

Defn A closed contour γ is a simple closed loop if $\forall z$ not on γ either $w(\gamma, z) = 0$ or 1.

If $w(\gamma, z) = 1$ then we say z is inside γ .



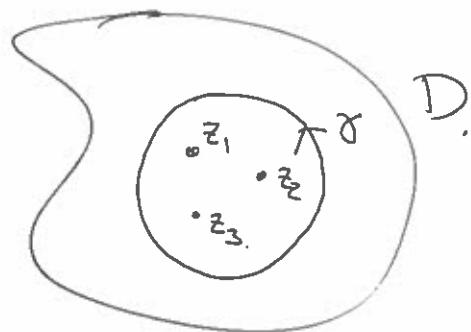
Cauchy's Residue Thm

Let D be a domain containing the simple closed loop γ and all the points inside γ .

Suppose $f: D \rightarrow \mathbb{C}$ is meromorphic with poles z_1, \dots, z_n inside γ .

Then

$$\oint_{\gamma} f = 2\pi i \sum_{j=1}^n \text{Res}(f, z_j).$$



Example Let $f(z) = \frac{2}{z-1} + 3 + 4(z-1)$.

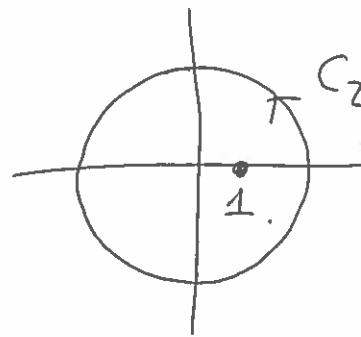
Then f has a simple pole at $z=1$ and $\text{Res}(f, 1) = 2$.

Let C_2 = circle centre O , radius 2, once anticlockwise.

By C.R.T.

$$\int_{C_2} f = 2\pi i \operatorname{Res}(f, 1)$$

$$= 2\pi i \times 2 = 4\pi i.$$



Example. $f(z) = \frac{1}{z^2 + (2-i)z - 2i} = \frac{1}{(z+2)(z-i)}$

f has simple poles at $z_0 = -2, z_0 = i$.

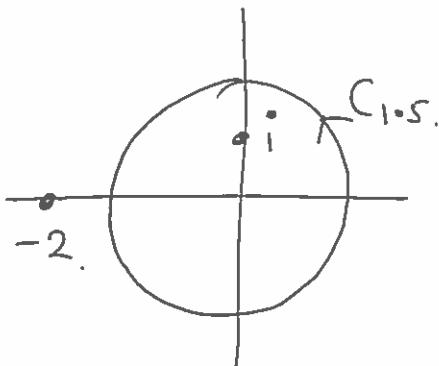
Recall: If z_0 is a simple pole then

$$\operatorname{Res}(f, z_0) = \lim_{z \rightarrow z_0} (z - z_0) f(z).$$

$$\operatorname{Res}(f, -2) = \lim_{z \rightarrow -2} (z+2) \times \frac{1}{(z+2)(z-i)} = \frac{-1}{2+i}.$$

$$\operatorname{Res}(f, i) = \lim_{z \rightarrow i} (z-i) \times \frac{1}{(z+2)(z-i)} = \frac{1}{2+i}.$$

Let $C_{1.5}$ be the circle, centre O , radius 1.5 , once anticlockwise.

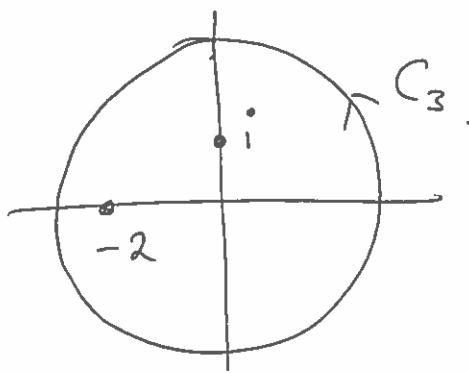


$$\int_{C_{1.5}} f = 2\pi i \operatorname{Res}(f, i) \text{ by C.R.T.}$$

$$= \frac{2\pi i}{2+i}$$

Let C_3 be the circle, centre O radius 3 , once anticlockwise.

(5)



By CRT

$$\int_{C_3} f = 2\pi i \left[\operatorname{Res}(f, i) + \operatorname{Res}(f, -2) \right]$$

$$= 2\pi i \left[\frac{1}{2+i} - \frac{1}{2+i} \right] = 0.$$

Applications to trig integrals

Use CRT to calculate integrals of the form

$$\int_0^{2\pi} Q(\cos t, \sin t) dt \quad (*) \quad (Q = \text{some function}).$$

$$\text{Eg: } \int_0^{2\pi} \left(\frac{\cos^4 t + \sin^2 t}{1 + \cos^3 t} \right) dt$$

Method: turn (*) into a complex integral

$$\text{Let } z = e^{it}$$

$$\cos t = \frac{e^{it} + e^{-it}}{2} = \frac{z + z^{-1}}{2}$$

$$\sin t = \frac{e^{it} - e^{-it}}{2i} = \frac{z - z^{-1}}{2i}$$

$$dz = ie^{it} dt = iz dt, \text{ so } dt = \frac{dz}{iz}.$$

As t varies from 0 to 2π , z travels around the circle C_1 = centre O , radius 1 , once anticlockwise. ⑥

So

$$\int_0^{2\pi} Q(\cos t, \sin t) dt = \int_{C_1} \left[Q\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right) \right] \frac{dz}{iz}.$$

$\underbrace{\hspace{10em}}$

can calculate this using CRT.
§

Example $\int_0^{2\pi} \sin^2 t dt.$

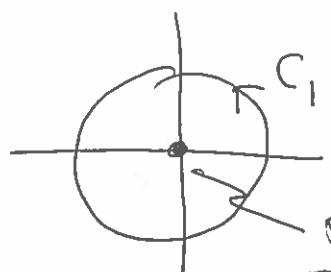
$$\text{Let } z = e^{it} \quad dt = \frac{dz}{iz}. \quad \sin t = \cancel{z+z^{-1}} \frac{z-z^{-1}}{2i}$$

$$\int_0^{2\pi} \sin^2 t dt = \int_{C_1} \left(\frac{z-z^{-1}}{2i} \right)^2 \cdot \frac{1}{iz} dz$$

$$= \int_{C_1} -\frac{z}{4i} + \frac{1}{2iz} - \frac{1}{4iz^3} dz.$$

pole of order 3?

residue at
 $z=0$ is $1/2i$.



pole at $z=0$
Residue = $1/2i$

$$\begin{aligned} \text{CRT.} &= 2\pi i \times \text{Residue at } 0 \\ &= 2\pi i \times \frac{1}{2i} = \pi. \end{aligned}$$