

What did we do last time?

①

• Stated Laurent's Thm:

" If f is holomorphic on an annulus then it can be expanded as a Laurent series on that annulus "

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad R_1 < |z-z_0| < R_2$$

• Defn f has a singularity at z_0 if f is not differentiable at z_0

• f not defined at $z_0 \Rightarrow f$ not diff'ble at z_0
 $\Rightarrow f$ has a singularity at z_0

What will we do today?

• Classify different types of singularities

Recall: A ~~disk~~ punctured disc $\{z \in \mathbb{C} \mid 0 < |z-z_0| < R\}$

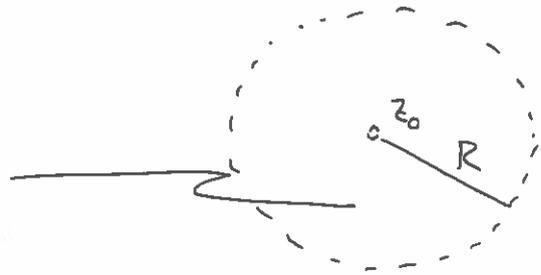
\ni an annulus.



Suppose f has a singularity at z_0 . ①

Defn Suppose \exists a punctured disc $\{z \in \mathbb{C} \mid 0 < |z - z_0| < R\}$ s.t. f is differentiable on this punctured disc. Then we say that f has an isolated singularity at z_0 .

f is diff'ble on this disc, except at z_0



Example $f(z) = 1/z$ is not defined at $z=0$, so cannot be differentiable at 0, so has a singularity at 0. Here 0 is an isolated singularity (as f is diff'ble on $0 < |z| < R$).

Suppose f has an isolated singularity at z_0 . Then f is holomorphic on the punctured disc $0 < |z - z_0| < R$ for some R & this is an annulus. By Laurent's Theorem we can expand f on a Laurent series on this punctured disc.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$

$$= \dots + \frac{b_n}{(z - z_0)^n} + \dots + \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

principal part of the Laurent series

$$(a_n, b_n \in \mathbb{C})$$

There are 3 possibilities for the principal part: ②

- ① the principal part has no terms in it.
- ② the principal part has finitely many terms in it.
- ③ the principal part has infinitely many terms in it.

Case ①: Removable singularity

Suppose f has an isolated singularity at z_0 & the principal part of the Laurent series has no terms in it. So for $0 < |z - z_0| < R$ we can expand f as a Laurent series

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots + a_n(z-z_0)^n + \dots$$

$a_n \in \mathbb{C}$ valid for $0 < |z - z_0| < R$

We can extend f to a differentiable function defined on $|z - z_0| < R$ by setting $f(z_0) = a_0$.

Example $f(z) = \frac{\sin z}{z}$, $z \neq 0$. This is not defined at the origin, so is not diff'ble at the origin, so has an isolated singularity at the origin.

Note: $f(z) = \frac{1}{z} \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$

Define $f(0) = 1$. Then f is diff'ble $\forall z \in \mathbb{C}$.

So f has a removable singularity at 0.

Case (3): Isolated essential singularities (3)

Suppose f has an isolated singularity at z_0 & the principal part of the Laurent series at z_0 valid on the punctured disc $0 < |z - z_0| < R$ has infinitely many terms in it. Then we say f has an isolated essential singularity at z_0 .

Example $f(z) = \cos 1/z$.

$\cos 1/z$ has a singularity at $z_0 = 0$ (because it's not defined at 0, so cannot be diff'ble at 0).

$$f(z) = \cos \frac{1}{z} = 1 - \frac{1}{2!z^2} + \frac{1}{4!z^4} - \frac{1}{6!z^6} + \dots$$

principal part.

As the principal part of the Laurent series contains infinitely many terms, the origin is an isolated essential singularity. Isolated essential singularities are HARD! (We'll ignore them!)

Case (2): Poles

Suppose f has an isolated singularity at z_0 & the principal part of the Laurent series for f at z_0 valid on ~~the~~ $0 < |z - z_0| < R$ has finitely many terms in it.

So for $0 < |z - z_0| < R$ we can write

$$f(z) = \frac{b_m}{(z - z_0)^m} + \dots + \frac{b_1}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad b_m \neq 0$$

We say f has a pole of order m at z_0 .

(Order = most negative power of $z - z_0$ that appears)

(9)

A pole of order 1 is called a simple pole.

Examples $f(z) = \frac{(\cos z) - 1}{z^4} \quad z \neq 0.$

Then f has an isolated singularity at 0.

$$\frac{\cos z - 1}{z^4} = \frac{1}{z^4} \left[\cancel{1} - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots - \cancel{1} \right]$$
$$= \frac{-1}{2! z^2} + \frac{1}{4!} - \frac{z^2}{6!} + \frac{z^4}{8!} - \dots$$

The most negative power is 2. So f has a pole of order 2 at the origin.

Defn $f: D \rightarrow \mathbb{C}$ is meromorphic if f is diff'ble on D except at finitely many points which are either removable singularities or poles.

Example $f(z) = 1/z$ is meromorphic on \mathbb{C} .

7. Cauchy's Residue Theorem

Zeros & poles of holomorphic functions

Let $f(z) = \frac{p(z)}{q(z)}$ & $q(z_0) = 0, p(z_0) \neq 0.$

Then f has a singularity at z_0 .

We want to link the zeros of q & the singularities of f .

What did we do last time?

⊙

- f has a singularity at z_0 if f is not differentiable at z_0
- f has an isolated singularity at z_0 if there are no other singularities nearby ($\exists \varepsilon > 0$ st f is diff'ble on $0 < |z - z_0| < \varepsilon$)
- classified isolated singularities:
 - removable singularities
 - poles of order m
 - isolated essential singularities.



- f has a pole of order m at z_0 if f can be expanded as a Laurent series

$$\frac{b_m}{(z - z_0)^m} + \dots + \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + \dots + a_n(z - z_0)^n + \dots$$

order

$$0 < |z - z_0| < R, \\ b_m \neq 0$$

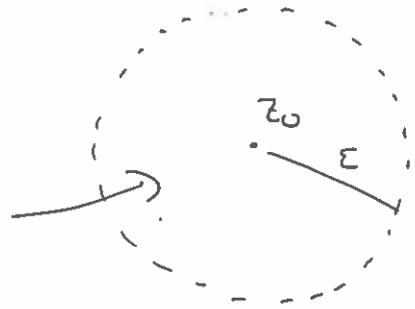
What will we do today?

- $f(z) = \frac{p(z)}{q(z)}$, $q(z_0) = 0$, $p(z_0) \neq 0$ has a singularity at z_0
- link zeros of q with poles of f .

Defn $f: D \rightarrow \mathbb{C}$ has an isolated zero at $z_0 \in D$ ①

if $f(z_0) = 0$ and $\exists \varepsilon > 0$ st $f(z) \neq 0$ for $0 < |z - z_0| < \varepsilon$.

$f(z_0) = 0$
no other zeros of f
in this disc



Suppose $f: D \rightarrow \mathbb{C}$ is holomorphic & has an isolated zero at $z_0 \in D$

Then on some disc centred at z_0 we can expand f as a

Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$= a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \dots + a_n (z - z_0)^n + \dots$$

Defn We say that f has a zero of order m at z_0 if:

$$a_0 = a_1 = \dots = a_{m-1} = 0, \quad a_m \neq 0.$$

We say z_0 is a simple zero if it has order 1.

Examples (1) $f(z) = z^3 = 0 + 0z + 0z^2 + 1 \cdot z^3$
- has a zero of order 3 at the origin.

$$(2) f(z) = (z-1)(z-3i)^4$$

- has a simple zero at $z=1$

- and a zero of order 4 at $z=3i$.

$$(3) f(z) = z^2 + 9 = (z+3i)(z-3i)$$

- has simple zeros at $z=3i, z=-3i$.

Remark Recall that if $f(z)$ has Taylor series at z_0 ②

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{then} \quad a_n = \frac{f^{(n)}(z_0)}{n!}$$

So f has a zero of order m at z_0 .

$$\Leftrightarrow a_0 = a_1 = \dots = a_{m-1} = 0 \quad a_m \neq 0$$

$$\Leftrightarrow f^{(k)}(z_0) = 0 \quad 0 \leq k \leq m-1 \quad f^{(m)}(z_0) \neq 0$$

In particular, f has a simple zero at z_0 if $f(z_0) = 0$, $f'(z_0) \neq 0$.

Examples

(1) $f(z) = \sin z$. Then f has zeros at $z = k\pi$, $k \in \mathbb{Z}$
 $f'(z) = \cos z$. So $f'(k\pi) = (-1)^k \neq 0$.
 So these zeros are simple.

(2) $f(z) = 1 - \cos z$. Then f has zeros at $z = 2k\pi$, $k \in \mathbb{Z}$
 $f'(z) = \sin z$ $f'(2k\pi) = 0$
 $f''(z) = \cos z$ $f''(2k\pi) = 1 \neq 0$.
 So these zeros have order 2.

Lemma Suppose f is holomorphic & has a zero of order m at z_0 . Then on some disc centred at z_0 we can write $f(z) = (z-z_0)^m g(z)$ where $g(z)$ is holomorphic and $g(z_0) \neq 0$.

PF As f has a zero of order m at z_0 , it has a (3)

Taylor series of the form

$$f(z) = a_m (z - z_0)^m + a_{m+1} (z - z_0)^{m+1} + \dots \quad a_m \neq 0.$$

$$= (z - z_0)^m \left[a_m + a_{m+1} (z - z_0) + \dots \right]$$

$$= (z - z_0)^m g(z), \quad g(z) = \sum_{n=0}^{\infty} a_{m+n} (z - z_0)^n$$

Then g is holomorphic on some disc centred at z_0 and $g(z_0) = a_m \neq 0$. □

Lemma Suppose $f(z) = \frac{p(z)}{q(z)}$ where

- p ~~is~~ is holomorphic, $p(z_0) \neq 0$
- q is holomorphic, q has a zero of order m at z_0 .

Then f has a pole of order m at z_0 .

Recall If f has a pole of order m at z_0 then f has a Laurent series

$$f(z) = \frac{b_m}{(z - z_0)^m} + \dots + \frac{b_1}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad b_m \neq 0$$

PF of lemma The assumption on q allows us to write $q(z) = (z - z_0)^m r(z)$ where r is holomorphic on some disc centred at z_0 and $r(z_0) \neq 0$.

Let $g(z) = \frac{p(z)}{r(z)}$. Then g is holomorphic on some disc centred at z_0 (note: $g(z_0) \neq 0$) & so we can expand it as a Taylor series.

$$g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{on some disc } |z - z_0| < R. \quad (9)$$

Note: $g(z_0) = \frac{p(z_0)}{r(z_0)} = a_0 \neq 0$.

Then $f(z) = \frac{p(z)}{q(z)} = \frac{p(z)}{(z - z_0)^m r(z)} = \frac{g(z)}{(z - z_0)^m}$

$$= \frac{a_0}{(z - z_0)^m} + \frac{a_1}{(z - z_0)^{m-1}} + \dots + \frac{a_{m-1}}{z - z_0} + a_m + a_{m+1}(z - z_0) + \dots$$

- a Laurent series where the most negative power of $(z - z_0)$ appearing is $(z - z_0)^m$ (as $a_0 \neq 0$)

So f has a pole of order m at z_0 . □

Examples

(1) $f(z) = \frac{\sin z}{(z-3)^2}$ - Find the poles!

$(z-3)^2$ has a zero of order 2 at $z_0 = 3$

$\sin 3 \neq 0$

So f has a pole of order 2 at $z_0 = 3$.

(2) $f(z) = \frac{z+3}{\sin z}$ - Find the poles!

$\sin z$ has a simple zero at $z_0 = k\pi$, $k \in \mathbb{Z}$

$z+3 \neq 0$ when $z_0 = k\pi$.

So f has simple poles at $z_0 = k\pi$, $k \in \mathbb{Z}$.