

What did we do last time?

- Estimation Lemma Suppose $|f(z)| \leq M \forall z \in \gamma$, a contour
Then $|\int_{\gamma} f| \leq M \times \text{length}(\gamma)$.
- Winding number formula: γ = closed contour, $z_0 \notin \gamma$.
Then $w(\gamma, z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - z_0}$
- Cauchy's Thm Suppose: $f: D \rightarrow \mathbb{C}$ holomorphic
 γ is a closed contour in D
 $w(\gamma, z) = 0 \quad \forall z \notin D$
Then $\int_{\gamma} f = 0$
- Generalised Cauchy Thm
Suppose: $f: D \rightarrow \mathbb{C}$ is holomorphic
 $\gamma_1, \dots, \gamma_n$ closed contours in D
 $w(\gamma_1, z) + \dots + w(\gamma_n, z) = 0 \quad \forall z \notin D$
Then $\int_{\gamma_1} f + \dots + \int_{\gamma_n} f = 0$

What will we do today?

- Prove the GCT
- Prove a (very surprising) result: Cauchy's Integral formula

Coursework test

- Next week (week 7!) - see your personalised timetables for time + location
- 40 mins, 2 (compulsory) questions
- What's examinable? Everything up to 8 including the Generalised Cauchy Theorem.

Proof of GCT

Suppose γ_j starts at z_j , $1 \leq j \leq n$.

Choose any point $z_0 \in D$. Choose contours $\sigma_1, \dots, \sigma_n$ in D s.t. σ_j starts at z_0 and ends at z_j .

Then $\sigma_j + \gamma_j - \sigma_j$ — this is a closed contour that starts & ends at z_0 . Note:

$$\omega(\sigma_j + \gamma_j - \sigma_j, z) = \omega(\gamma_j, z) \quad \forall z \notin D$$

$$\text{Let } \gamma = \sigma_1 + \gamma_1 - \sigma_1 + \sigma_2 + \gamma_2 - \sigma_2 + \dots + \sigma_n + \gamma_n - \sigma_n$$

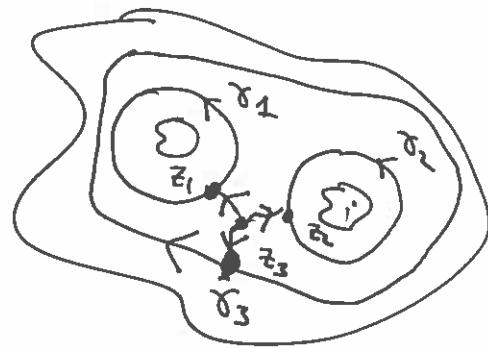
Then γ is a closed loop that starts & ends at z_0

Apply Cauchy's Thm to γ : Let $z \notin D$. Then

$$\omega(\gamma, z) = \sum_{j=1}^n \omega(\sigma_j + \gamma_j - \sigma_j, z) = \sum_{j=1}^n \omega(\gamma_j, z) = 0 \quad \text{by hypothesis.}$$

So, by Cauchy's Thm,

$$\begin{aligned} 0 = \int_{\gamma} f &= \int_{\sigma_1 + \gamma_1 - \sigma_1 + \dots + \sigma_n + \gamma_n - \sigma_n} f = \sum_{j=1}^n \left(\int_{\sigma_j} f + \int_{\gamma_j} f + \int_{-\sigma_j} f \right) \\ &= \sum_{j=1}^n \int_{\gamma_j} f \quad \left(\text{as } \int_{-\sigma_j} f = - \int_{\sigma_j} f \right). \end{aligned} \quad \square.$$

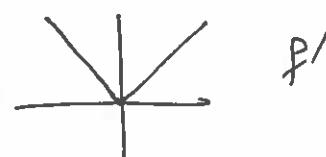


§5 Cauchy's Integral formula + Taylor's Thm

In real analysis, some functions can be differentiated once but not twice, eg: $f(x) = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$



$$f'(x) = \begin{cases} 2x & x \geq 0 \\ -2x & x < 0 \end{cases}$$



f' is not differentiable at 0.

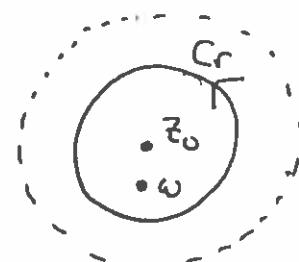
In complex analysis, this can't happen! If f can be differentiated once on a domain then it can be differentiated infinitely many times. ②

Thm (Cauchy's Integral formula)

Suppose: f is holomorphic on the disc $\{z \in \mathbb{C} \mid |z - z_0| < R\}$

C_r = circle centre z_0 radius $r < R$, described once anti-clockwise
 w be s.t. $|w - z_0| < r$

Then: $f(w) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z-w} dz.$



PP (not examinable - but please still listen/pay attention).

Fix w s.t. $|w - z_0| < r$. Consider $g(z) := \frac{f(z) - f(w)}{z-w}$.

Then g is holomorphic on the domain

$$D = \{z \in \mathbb{C} \mid |z - z_0| < R, z \neq w\}.$$

Let $\varepsilon > 0$. Let S_ε be the circle centre w radius ε , once anti-clockwise

Note $C_r, S_\varepsilon \subset D$ (if ε small enough)

Apply GCT to $S_\varepsilon, -C_r$

Suppose $z \notin D$. There are two cases:

$$\textcircled{1}: |z - z_0| > R \quad \omega(S_\varepsilon, z) + \omega(-C_r, z) = 0 + 0 = 0$$

$$\textcircled{2}: z = w \quad \omega(S_\varepsilon, z) + \omega(-C_r, z) = +1 + (-1) = 0.$$

Hence $\omega(S_\varepsilon, z) + \omega(-C_r, z) = 0 \quad \forall z \notin D.$

By the GCT $\int_{\Sigma} g + \int_{-C_r} g = 0.$ (3) (4)

Hence $\int_{S_\varepsilon} g = - \int_{-C_r} g = \int_{C_r} g.$

Recall $g(z) = \frac{f(z) - f(w)}{z-w}$. Hence $\lim_{z \rightarrow w} g(z) = f'(w).$

As $f'(w)$ is finite, it follows that $g(z)$ is bounded for z near w .
i.e. $\exists \delta > 0 \exists M > 0$ s.t. if $0 < |z-w| < \delta$ then $|g(z)| \leq M.$

Choose $\varepsilon < \delta$. By the Estimation Lemma

$$\left| \int_{S_\varepsilon} g \right| \leq M \times \text{length}(S_\varepsilon) = 2\pi M \varepsilon \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

As $\underbrace{\int_{C_r} g}_{\Sigma} = \int_{S_\varepsilon} g \rightarrow 0 \text{ as } \varepsilon \rightarrow 0$

does not depend on ε .

i.e. $\int_{C_r} \frac{f(z) - f(w)}{z-w} dz = 0$

$$\begin{aligned} \int_{C_r} \frac{f(z)}{z-w} dz &= f(w) \int_{C_r} \frac{dz}{z-w} && \text{winding number} \\ &= f(w) \times 2\pi i \omega(C_r, w) && \text{of } C_r \text{ around } w \\ &= 2\pi i f(w). && \downarrow = 1. \end{aligned}$$

□.

Taylor's Thm

Suppose: f is holomorphic on D

Then: • all the higher derivatives of f exist on D

• on any disc $\{z \in \mathbb{C} \mid |z - z_0| < R\} \subset D$

f is equal to its Taylor series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

Moreover, if C_r = circle centre z_0 , radius r ($r < R$)
once anti-clockwise

then $f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{C_r} \frac{f(z)}{(z - z_0)^{n+1}} dz$.

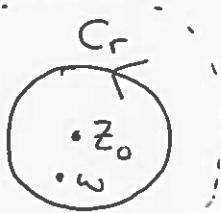
What did we do last time?

- Cauchy's Integral Formula

Suppose: f is holomorphic on the disc $\{z \in \mathbb{C} \mid |z - z_0| < R\}$

C_r = circle, centre z_0 , radius $r < R$, described once anticlockwise

w s.t. $|w - z_0| < r$



Then $f(w) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z - w} dz$

- Taylor's Thm

Suppose: f is holomorphic on a domain D

- Then:
- all the higher derivatives of f exist on D
 - on any disc $\{z \in \mathbb{C} \mid |z - z_0| < R\} \subset D$

f has a Taylor series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

Moreover: if C_r = circle, centre z_0 , radius r ($r < R$)
once anticlockwise

then • $f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_r} \frac{f(z)}{(z - z_0)^{n+1}} dz$

What will we do today?

- Prove:
- Cauchy's Estimate
 - The Fundamental Thm of Algebra
 - Liouville's Thm

(Basically, we're going to do lots of proofs ☺)

Defn A function f is analytic if it is equal to its ① Taylor Series.

In real analysis: some functions can have a Taylor series but not be equal to the Taylor Series

$$\text{eg: } f(x) = \begin{cases} e^{-\frac{1}{2}x^2} & x \neq 0 \\ 0 & x = 0 \end{cases} \quad f: \mathbb{R} \rightarrow \mathbb{R}$$



$$\text{Here } f^{(n)}(0) = 0 \quad \forall n.$$

So f has Taylor series 0 at the origin.

But f is not identically equal to 0.

In complex analysis, every holomorphic function is analytic.

Cauchy's Estimate

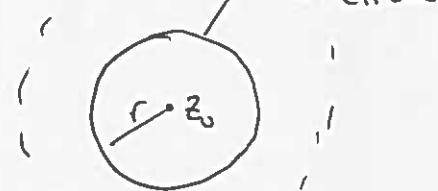
Suppose: f is holomorphic on $\{z \in \mathbb{C} \mid |z - z_0| < R\}$

$$0 < r < R$$

$$|f(z)| \leq M \quad \forall z \text{ st. } |z - z_0| = r$$

$$\text{Then: } \forall n \geq 0 \quad |f^{(n)}(z_0)| \leq \frac{M}{r^n} n!$$

| know
| $|f(z)| \leq M$
on this
circle



Pf Recall the Estimation Lemma:

$$\boxed{\int_{\gamma} f(z) dz} \leq (\text{bound on } |f(z)| \text{ for points } z \in \gamma) \times \text{length}(\gamma).$$

By Taylor's Thm

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi} \left| \int_{C_r} \frac{f(z)}{(z - z_0)^{n+1}} dz \right|$$

C_r = circular contour centre z_0
radius r
once anticlockwise.

$$\text{Let } z \in C_r. \text{ Then } |f(z)| \leq M, \quad |z - z_0|^{n+1} = r^{n+1}.$$

$$\text{Hence if } z \in C_r \text{ then } \left| \frac{f(z)}{(z-z_0)^{n+1}} \right| \leq \frac{M}{r^{n+1}}. \quad (2)$$

Clearly length (C_r) = $2\pi r$.

By the Estimation Lemma

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \times \frac{M}{r^{n+1}} \times 2\pi r = \frac{M \cdot n!}{r^n}. \quad \square$$

Liouville Thm

Suppose $f: \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic and bounded
 \downarrow
 ie $\exists M > 0$ st $|f(z)| \leq M \forall z \in \mathbb{C}$.

Then f is a constant.

Recall: If $f'(z_0) = 0 \forall z_0 \in D$ then f is constant on D .

Pf of Liouville Choose M st $|f(z)| \leq M \forall z \in \mathbb{C}$. Let $z_0 \in \mathbb{C}$.
 We show $f'(z_0) = 0$. As f is holomorphic on \mathbb{C} , it follows
 that f is holomorphic on $\{z \in \mathbb{C} \mid |z - z_0| < R\}$ for any $R > 0$.

Apply Cauchy's Estimate with $n=1$: Then for any $0 < r < R$
 we have $|f'(z_0)| \leq \frac{M}{r}$.

Take R , then r , as large as we please then we get

$|f'(z_0)| = 0$. So $f'(z_0) = 0$. Hence $f = \text{constant}$ \square .

Fundamental Thm of Algebra

Motivation:	solve	$x - n = 0$	- solutions in \mathbb{N}
	solve	$x + n = 0$	- solution in \mathbb{Z}
	solve	$px - q = 0$	- solutions in \mathbb{Q}
	solve	$x^2 - 2 = 0$	- solutions in \mathbb{R}
	solve	$x^2 + 1 = 0$	- solutions in \mathbb{C} .

What about polynomials with complex coefficients?

e.g. can we solve $z^2 - i = 0$ in \mathbb{C} ?

Thm (Fund. Thm. of Algebra) Let $p(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0$ be a degree $n \geq 1$ polynomial with coefficients $a_j \in \mathbb{C}$. Then $\exists \alpha \in \mathbb{C}$ st. $p(\alpha) = 0$.

Corollary Let $p(z)$ be as above. Then we can factorise

$$p(z) = \prod_{j=1}^n (z - \alpha_j) \text{ for some } \alpha_1, \dots, \alpha_n \in \mathbb{C}.$$

Thm \Rightarrow Cor Let $p(z)$ be a degree n polynomial. By the Fund. Thm. of Algebra we can $\alpha_i \in \mathbb{C}$ s.t. $p(\alpha_i) = 0$. Write $p(z) = (z - \alpha_1)q(z)$ where q is a degree $n-1$ polynomial with complex coefficients. Now induct on n . \square .

Pf of Fund. Thm of Algebra Suppose $p(z)$ does NOT have a root, ie $p(z) \neq 0 \quad \forall z \in \mathbb{C}$.

Hence $\frac{1}{p(z)}$ is holomorphic $\mathbb{C} \rightarrow \mathbb{C}$.

By Liouville's Thm, if we can show $\frac{1}{p(z)}$ is bounded ④
 then it will follow that $\frac{1}{p(z)}$ is constant, a contradiction.

We show $\left| \frac{1}{p(z)} \right| \leq M \quad \forall z \in \mathbb{C}$.

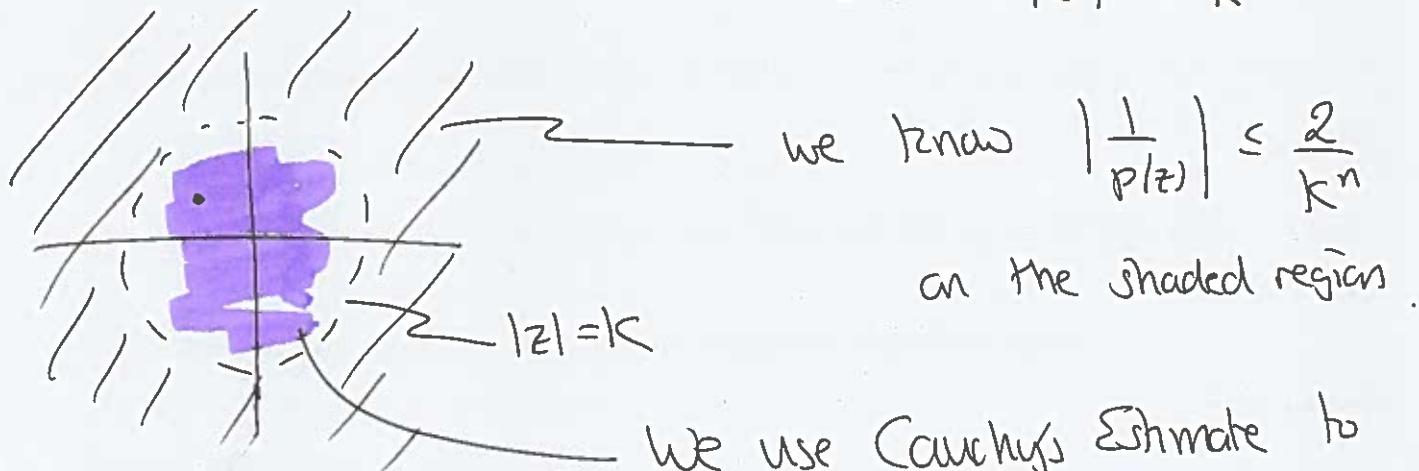
For $z \neq 0$

$$\frac{p(z)}{z^n} = 1 + \frac{a_{n-1}}{z} + \dots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}$$

Note: $\frac{p(z)}{z^n} \rightarrow 1$ as $z \rightarrow \infty$.

Hence $\exists k$ st if $|z| > k$ then $\left| \frac{p(z)}{z^n} \right| \geq \frac{1}{2} \cdot \frac{1}{2}$.

Rearrange this: if $|z| > k$ then $\left| \frac{1}{p(z)} \right| \leq \frac{2}{|z|^n} \leq \frac{2}{k^n}$.



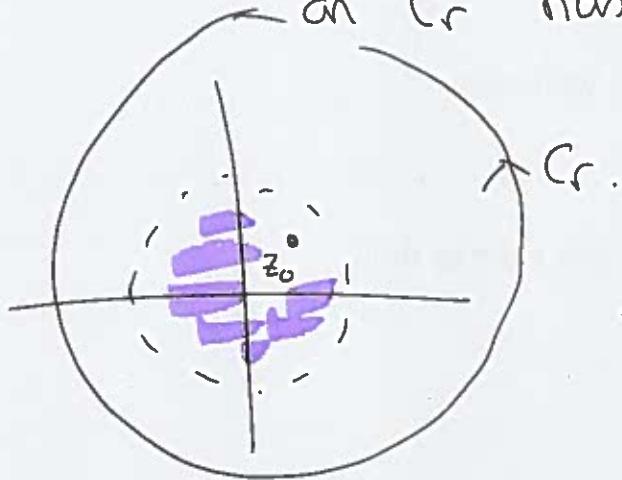
We use Cauchy's Estimate to
 show $\left| \oint \frac{1}{p(z)} \right| \leq \frac{2}{k^n}$ here as well

(5)

Let $z_0 \in \mathbb{C}$ & suppose $|z_0| < k$.

Let C_r = circular path centre z_0 , radius r ,
once anticlockwise

where r is large so that every point
on C_r has modulus $\geq k$.



By Cauchy's Estimate
(in the case $n=0$) we
have $\left| \frac{1}{p(z_0)} \right| \leq \frac{2}{k^n}$.

Hence $\left| \frac{1}{p(z)} \right| \leq \frac{2}{k^n} \quad \forall z \in \mathbb{C}$.

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