What did we do last time?

- Defined the (complex) exponential function

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad (\text{converges } \forall z \in \mathbb{C})$$

- Defined the complex logarithm: for $z \in \mathbb{C} \setminus \{0\}$

$$\log z := \ln|z| + i \arg(z) \quad (\text{a multivalued function})$$

and the principal value of the complex log:

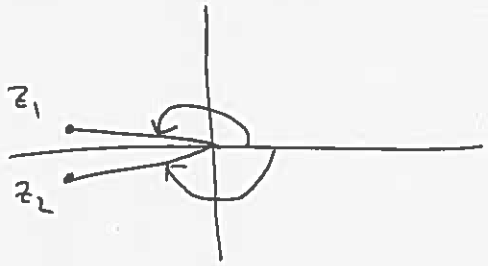
$$\operatorname{Log} z := \ln|z| + i \underbrace{\operatorname{Arg}(z)}_{\in (-\pi, \pi]}.$$

What will we do today?

- study the differentiability of $\operatorname{Log} z$
- start looking at the complex integral by defining paths, loops and contours.

The principal value of the complex log, $\mathrm{Log}\, z$, ①
is not continuous (& therefore not differentiable)
on $\mathbb{C} \setminus \{0\}$.

Why? Because $\mathrm{Arg}(z)$ is not continuous on $\mathbb{C} \setminus \{0\}$.



$\mathrm{Arg}(z_1)$ is close to π
$\mathrm{Arg}(z_2)$ is close to $-\pi$.
As we go across the negative real axis, Arg jumps from π to $-\pi$, so is not continuous.

Define the <u>cut plane</u> $D := \mathbb{C} \setminus \left\{ \begin{array}{l} \text{negative real axis} \\ \text{and } 0 \end{array} \right\}$.

<u>Fact</u> $\mathrm{Log}\, z$ is continuous on the cut-plane D.
Why? Recall $\mathrm{Log}\, z = \ln|z| + i\,\mathrm{Arg}(z)$.
and note that $|z|$, $\mathrm{Arg}(z)$, $\ln$ are all continuous.

<u>Prop</u> $\mathrm{Log}\, z$ is holomorphic on the cut-plane D and

$$\frac{d}{dz} \mathrm{Log}\, z = \frac{1}{z}.$$

<u>Pf</u> Let $w = \mathrm{Log}\, z$, so $z = \exp(w)$. Let $\mathrm{Log}(z+h) = w + k$.
As $\mathrm{Log}\, z$ is continuous on D, we have $k \to 0$ as $h \to 0$

$$\mathrm{Log}'(z) = \lim_{h \to 0} \frac{\mathrm{Log}(z+h) - \mathrm{Log}\, z}{(z+h) - z}$$

$$= \lim_{k \to 0} \frac{(w+k) - w}{\exp(w+k) - \exp(w)}$$

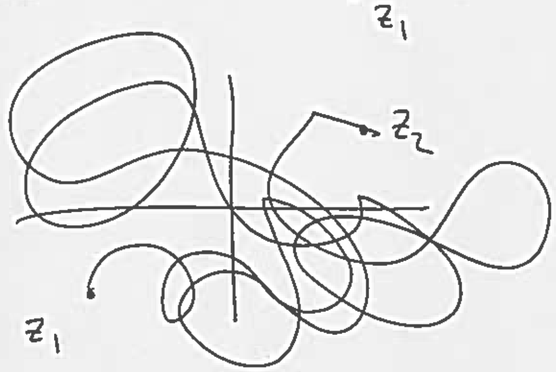
$$= \lim_{h\to 0}\left(\frac{\exp(w+h)-\exp(w)}{h}\right)^{-1}$$

$$= \left(\exp'(w)\right)^{-1} = \frac{1}{\exp(w)} = \frac{1}{z} \qquad \square$$

4. Complex integration & Cauchy's Thm

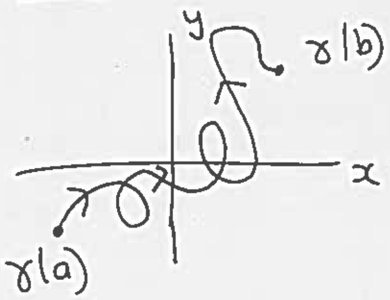
Real case: $\int_a^b f(x)\,dx$ = "integral of f from a to b"

What does $\int_{z_1}^{z_2} f(z)\,dz$ mean?



Need to look at paths γ from z_1 to z_2.

Defn A path is a continuous function $\gamma : [a,b] \to \mathbb{C}$ ($[a,b] \subset \mathbb{R}$ is an interval). The path starts at $\gamma(a)$ and ends at $\gamma(b)$.



$a \le t \le b$

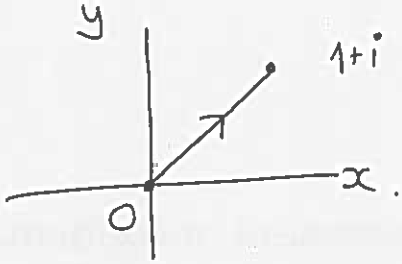
Think of $\gamma(t)$ as "where we are in the complex plane at time t".

~~Defn~~ Rmk γ is a function. Sometimes it is useful to regard γ as a set of points in $\mathbb{C}$ (ie we identify γ with its image), albeit equipped with an orientation.

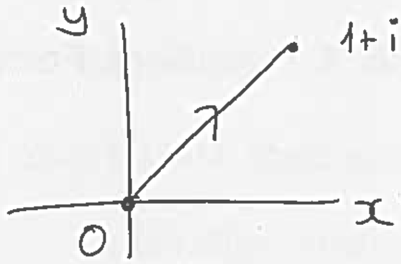
Often we call the function $\gamma(t)$, $a \le t \le b$, a parametrisation of γ

Different parametrisations can give the same path

$\gamma(t) = t + it \quad 0 \le t \le 1$

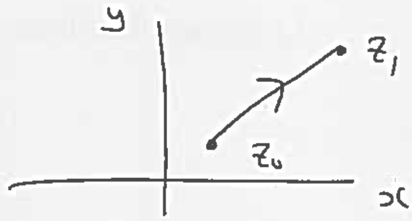


$\gamma(t) = t^2 + it^2 \quad 0 \le t \le 1.$



__Examples of paths__ Let $z_0, z_1 \in \mathbb{C}$. Let

$\gamma(t) = (1-t)z_0 + tz_1 \quad 0 \le t \le 1.$

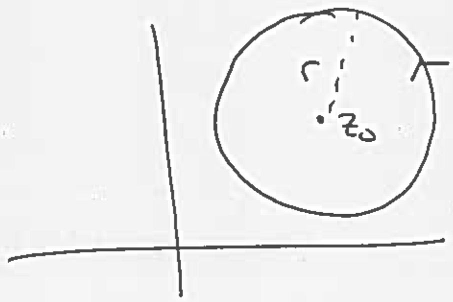


- a straight line from z_0 to z_1
- Denote this by $[z_0, z_1]$

__Defn__ γ is a __closed loop__ if it starts & ends at the same point: $\gamma : [a,b] \to \mathbb{C} \quad \gamma(a) = \gamma(b)$

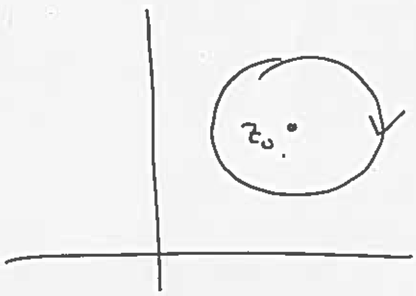
__Example__ Let $z_0 \in \mathbb{C}$, $r > 0$

$\gamma(t) = z_0 + re^{it} \quad 0 \le t \le 2\pi$



- describes a circle, centre z_0 radius r, once anti-clockwise

$\gamma(t) = z_0 + re^{-it} \quad 0 \le t \le 2\pi.$



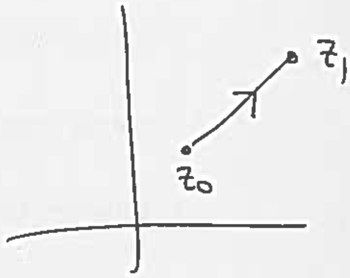
- circle centre z_0 radius r once clockwise

Defn γ is <u>smooth</u> if $\gamma(t)$ is differentiable and $\gamma'(t)$ is continuous (means: γ has no corners).

Defn Let $\gamma : [a,b] \to \mathbb{C}$ be a smooth path. Then

$$\text{length}(\gamma) := \int_a^b |\gamma'(t)|\, dt$$

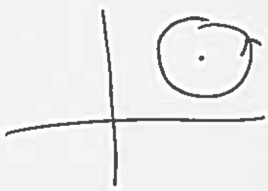
Examples: (1) $\text{length}\,[z_0, z_1] = |z_1 - z_0|$



(2) $\gamma(t) = z_0 + re^{it} \quad 0 \le t \le 2\pi$

$\text{length}(\gamma) = 2\pi r$.

(easy checks from the defn).

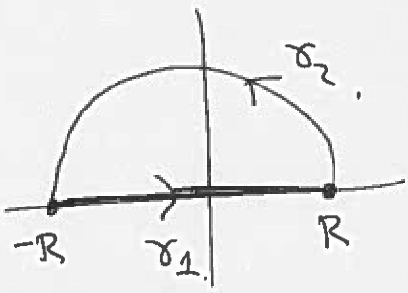


Defn A <u>contour</u> γ is a collection of smooth paths $\gamma_1, \dots, \gamma_n$ s.t. the end point of γ_r is the start point of γ_{r+1}. (Essentially: a contour is a smooth path, except we allow finitely many corners.)

Write $\gamma = \gamma_1 + \gamma_2 + \dots + \gamma_n$.

Example $\gamma_1(t) = t \quad -R \le t \le R$

$\gamma_2(t) = Re^{it} \quad 0 \le t \le \pi$

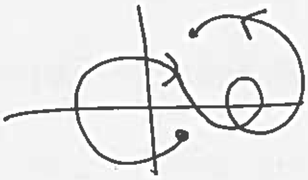


$\gamma = \gamma_1 + \gamma_2$ is a D-shaped contour.

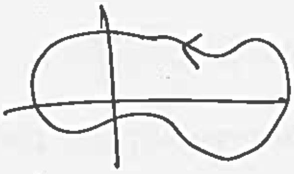
Defn γ is a <u>closed contour</u> if it starts & ends at the same point.

What did we do last time?

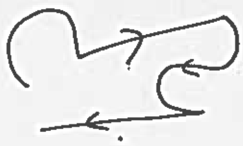
- we proved that $\operatorname{Log} z$ is holomorphic on the cut plane and $\frac{d}{dz} \operatorname{Log} z = \frac{1}{z}$
- we defined paths



closed loops



contours.

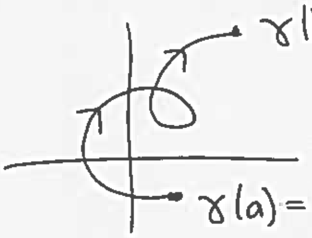


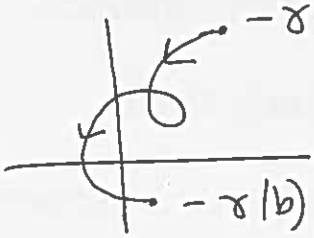
What will we do today?

- define the reversed path $-\gamma$
- define the complex integral $\int_\gamma f$
- state & prove the Fundamental Thm of Contour Integration.

Reversed paths Let $\gamma : [a,b] \to \mathbb{C}$ be a path.

The reversed path is $-\gamma : [a,b] \to \mathbb{C}$ is defined to be $-\gamma(t) := \gamma(a+b-t)$



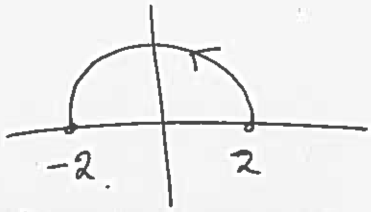


γ starts at z_0 & ends at z_1

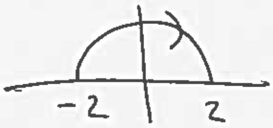
$-\gamma$ starts at z_1, travels backwards along γ, & ends at z_0

This **DOES NOT** mean: take the formula for γ and multiply it by -1

Example $\gamma(t) = 2e^{it}$ $0 \le t \le \pi$



$$-\gamma(t) = \gamma(0 + \pi - t) = 2e^{i(\pi - t)} = -2e^{-it} \quad 0 \le t \le \pi$$



Contour integration Let $f : D \to \mathbb{C}$ be a function, $\gamma : [a,b] \to D$ be a smooth path in D.

We define: $$\int_\gamma f = \int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

Example Let $f(z) = z^2$, $\gamma(t) = t^2 + it^3$ $1 \le t \le 2$.

$\gamma'(t) = 2t + 3it^2$

$f(\gamma(t)) = (t^2 + it^3)^2 = t^4 + 2it^5 - t^6$

$$\int_\gamma f = \int_1^2 f(\gamma(t))\,\gamma'(t)\,dt = \int_1^2 (t^4 + 2i t^5 - t^6)(2t + 3it^2)\,dt$$

$$= \int_1^2 -8t^7 + 2t^5\,dt + i\int_1^2 7t^6 - 3t^8\,dt$$

$$= -234 - \frac{130i}{3}.$$

<u>Defn</u> Let $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$ be a contour. Then

$$\int_\gamma f := \int_{\gamma_1} f + \cdots + \int_{\gamma_n} f.$$

<u>Prop</u> "Contour integration is independent of the choice of parametrisation."

Let $\gamma : [a,b] \to \mathbb{C}$ be a smooth path

$\phi : [c,d] \to [a,b]$ is an increasing smooth function & a bijection.

(so $\gamma \circ \phi : [c,d] \to \mathbb{C}$ is a "reparametrisation" of γ)

Then $$\int_{\gamma \circ \phi} f = \int_\gamma f.$$

<u>Pf</u> Omitted – chain rule + change of variables.

<u>The algebra of contour integration</u>

Let $f, g : D \to \mathbb{C}$, $c \in \mathbb{C}$, $\gamma, \gamma_1, \gamma_2$ are contours in D (end point of γ_1 = start point of γ_2).

Then (1) $\displaystyle\int_{\gamma_1+\gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f$

(2) $\displaystyle\int_\gamma f+g = \int_\gamma f + \int_\gamma g$

(3) $\displaystyle\int_\gamma cf = c\int_\gamma f$

(4) $\displaystyle\int_{-\gamma} f = -\int_\gamma f$.

In real analysis, integration is the reverse of differentiation. To calculate $\int_a^b f(x)\,dx$, choose F s.t. $F' = f$, then

$\int_a^b f(x)\,dx = F(b) - F(a)$.

In complex analysis, <u>very few</u> functions have anti-derivatives (but when they do, calculating $\int_\gamma f$ is easy).

<u>Thm</u> (Fundamental Theorem of Contour Integration)

Suppose $g : D \to \mathbb{C}$ is continuous

$G : D \to \mathbb{C}$ is an anti-derivative of g on D

(ie $G'(z) = g(z)$ $\forall z \in D$).

γ is a contour in D from z_0 to z_1.

Then $\displaystyle\int_\gamma g = G(z_1) - G(z_0)$.

<u>Pf</u> It's sufficient to prove this for a smooth path γ.

Let $\gamma : [a,b] \to D$ be s.t. $\gamma(a) = z_0$, $\gamma(b) = z_1$.

Let $\omega(t) = g(\gamma(t))\,\gamma'(t)$

$W(t) = G(\gamma(t))$.

Then $W'(t) = G'(\gamma(t))\,\gamma'(t)$ (chain rule)

$= g(\gamma(t))\,\gamma'(t) = \omega(t)$.

Write $\omega(t) = u(t) + iv(t)$

$W(t) = U(t) + iV(t)$, so $U'(t) = u(t)$, $V'(t) = v(t)$

Hence: $\int_\gamma g = \int_a^b g(\gamma(t))\gamma'(t)\,dt = \int_a^b \omega(t)\,dt$

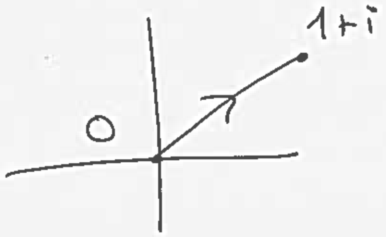
$= \int_a^b u(t) + iv(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$

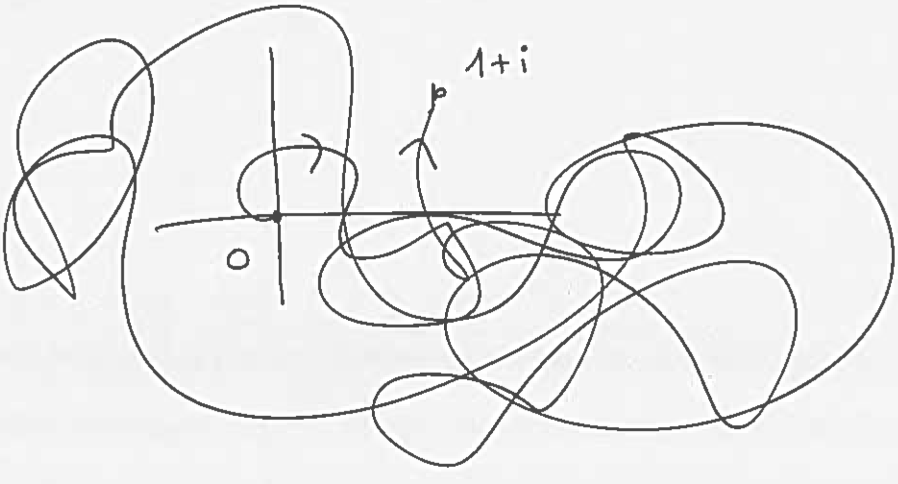
$= U(t)\Big|_a^b + i\,V(t)\Big|_a^b$ by Fund. Th. of Calculus

$= W(t)\Big|_a^b = G(\gamma(b)) - G(\gamma(a)) = G(z_1) - G(z_0)$ $\square$.

Rmk Suppose f has an antiderivative on D. Let γ be a contour in D. Then $\int_\gamma f$ depends only on the endpoints of γ & not on the choice of γ.

Example Let $f(z) = z^4$, γ = any contour from 0 to $1+i$.





Calculate $\int_\gamma f$.

Let $F(z) = \frac{1}{5} z^5$. Then $F'(z) = f(z)$ $\forall z \in \mathbb{C}$.

By the Fund. Thm of Contour Integration

$$\int_\gamma f = F(1+i) - F(0) = \frac{(1+i)^5}{5} - 0 = \frac{-4-4i}{5}$$

Rmk Suppose γ is a closed contour (ie starts & ends at the same point) & f has an anti-derivative on a domain that contains γ.

Then $\int_\gamma f = 0$.

[Why? If $F'(z) = f(z)$ then $\int_\gamma f = F(z_1) - F(z_0) = 0$ if $z_1 = z_0$.]