What did we do last time?

- defined what it meant for $f: D \to \mathbb{C}$ to be holomorphic
- stated the <u>Cauchy–Riemann Theorem</u>:
  Suppose $f: D \to \mathbb{C}$ and write $f(x+iy) = u(x,y) + iv(x,y)$
  Let $z_0 \in D$. Suppose f is diff'ble at $z_0 = x_0 + iy_0$.
  Then

  (1) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist at (x_0, y_0)

  (2) the Cauchy-Riemann eqns hold at (x_0, y_0)

  $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

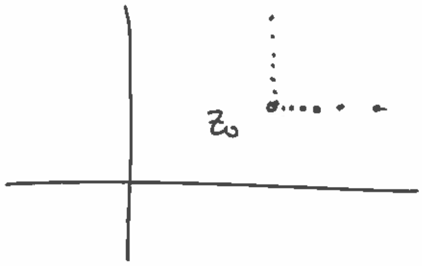
What will we do today?

- prove the Cauchy-Riemann Thm.
- see how to use the Cauchy-Riemann Thm (& its partial converse) to decide if a function is differentiable.

Proof of the Cauchy-Riemann Thm

Recall $f'(z_0) := \lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$.

Idea



different ways ~~for~~ for z to approach z_0 - we want to get the same limit.

Calculate $f'(z_0)$ in two ways:

Let $h \in \mathbb{R}$, $z = z_0 + h = (x_0 + h) + iy_0$. So $z \to z_0$ as $h \to 0$.

$$f'(z_0) = \lim_{h \to 0} \frac{[u(x_0+h, y_0) + iv(x_0+h, y_0)] - [u(x_0, y_0) + iv(x_0, y_0)]}{(x_0 + h + iy_0) - (x_0 + iy_0)}.$$

$$= \lim_{h \to 0} \left[\frac{u(x_0+h, y_0) - u(x_0, y_0)}{h}\right] + i\left[\frac{v(x_0+h, y_0) - v(x_0, y_0)}{h}\right]$$

$$= \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0).$$

Let $k \in \mathbb{R}$, $z = z_0 + ik = x_0 + i(y_0 + k)$. So $z \to z_0$ as $k \to 0$.

$$f'(z_0) = \lim_{k \to 0} \frac{[u(x_0, y_0+k) + iv(x_0, y_0+k)] - [u(x_0, y_0) + iv(x_0, y_0)]}{(x_0 + i(y_0 + k)) - (x_0 + iy_0)}$$

$$= \lim_{k \to 0} \left[\frac{u(x_0, y_0+k) - u(x_0, y_0)}{ik}\right] + i\left[\frac{v(x_0, y_0+k) - v(x_0, y_0)}{ik}\right]$$

$$= \lim_{k \to 0} \left[\frac{v(x_0, y_0+k) - v(x_0, y_0)}{k}\right] - i\left[\frac{u(x_0, y_0+k) - u(x_0, y_0)}{k}\right]$$

$$= \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0).$$

Hence the partial derivatives of u, v exist at (x_0, y_0).

Comparing the real & imag. parts of the two expressions for $f'(z_0)$ gives:

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0), \quad \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0).$$

$\square$

<u>Example</u> Let $f: \mathbb{C} \to \mathbb{C}$, $f(z) = \bar{z}$. Show that f is not differentiable at any point in $\mathbb{C}$.

Write $f(x+iy) = x - iy$, $u(x,y) = x$, $v(x,y) = -y$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = -1.$$

There are no points $x + iy \in \mathbb{C}$ at which $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$.

So there are no points in $\mathbb{C}$ at which the C-R eqns hold.

So there are no points in $\mathbb{C}$ at which f is differentiable.

Be <u>very</u> careful with the logic here!

The C-R Thm says:

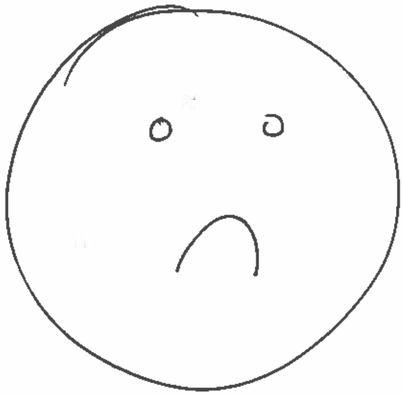
<u>IF</u> f is diff'ble at z_0 <u>THEN</u>
- partial derivs exist at (x_0, y_0)
- C-R eqns hold.

Is the converse true?

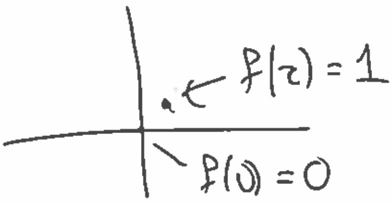
<u>IF</u>
- partial derivs exist at (x_0, y_0)
- C-R eqns hold

<u>IS IT TRUE THAT</u> f is diff'ble at z_0?

NO!



Example Define $f(x+iy) = \begin{cases} 0 & \text{if } x+iy \in \text{real axis or imag axis.} \\ 1 & \text{otherwise.} \end{cases}$



Note: f is not continuous at 0 because $1 = f(h+ih) \not\to f(0) = 0$ as $h \to 0$.

So f is not diff'ble at 0.

However, write $f(x+iy) = u(x,y) + iv(x,y)$. Then

$$\frac{\partial u}{\partial x}(0,0) = \lim_{h\to 0} \frac{u(h,0) - u(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = \lim_{h\to 0} 0 = 0$$

Similarly $\frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are all equal to 0 at the origin.

So the partial derivs exist at the origin, & the C-R eqns hold at the origin. But f is not diff'ble at the origin.

<u>Prop</u> (Partial converse to the C-R Thm).

Let $f: D \to \mathbb{C}$ be continuous. Write $f(x+iy) = u(x,y) + iv(x,y)$.

Let $z_0 = x_0 + iy_0 \in D$.

Suppose:
- partial derivs of u,v exist at (x_0, y_0)
- partial derivs of u,v are continuous at (x_0,y_0)
- C-R eqns to hold at (x_0,y_0).

Then:
- f is diff'ble at z_0.

<u>Pf:</u> see notes

---

§3. Power series & elementary functions

Recall: let $s_n \in \mathbb{C}$, $s \in \mathbb{C}$. We say $s_n \to s$ as $n \to \infty$ if:
$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ st $\forall n \geq N$ $|s_n - s| < \varepsilon$.

Let $z_k \in \mathbb{C}$. We say that the series $\sum_{k=0}^{\infty} z_k$ <u>converges</u>
if the sequence $s_n := \sum_{k=0}^{n} z_k$ converges.

We call the limit $\sum_{k=0}^{\infty} z_k$ the sum of the series.

A series that does not converge is called <u>divergent</u>.

What did we do last time?

- proved the Cauchy-Riemann Thm
- saw how to use the C-R Thm to show a function is not differentiable.
- stated the partial converse to the C-R Thm
- defined what $\sum_{n=0}^{\infty} z_n$ means.

What will we do today?

- state some basic properties of series
- use the ratio test / root test to decide convergence
- introduce power series: functions defined by series.

Rmk Write $z_n = x_n + iy_n$. Then

$\sum_{n=0}^{\infty} z_n$ converges $\iff$ $\sum_{n=0}^{\infty} x_n$, $\sum_{n=0}^{\infty} y_n$ both converge.

We need a stronger property:

Defn $\sum_{n=0}^{\infty} z_n$ is absolutely convergent if $\sum_{n=0}^{\infty} |z_n|$ converges.

Lemma Suppose $\sum_{n=0}^{\infty} z_n$ is absolutely convergent. Then $\sum_{n=0}^{\infty} z_n$ converges.

Rmk The converse is not true: convgt $\not\Rightarrow$ abs. convgt.

Example $z_n = \frac{(-1)^n}{n}$. Then $\sum_{n=0}^{\infty} z_n$ converges, but $\sum_{n=0}^{\infty} |z_n|$ diverges.

Multiplying series together

Suppose we have two series $\sum a_n$, $\sum b_n$. How do we multiply them together?

$$(a_0 + a_1 + a_2 + a_3 + \cdots)(b_0 + b_1 + b_2 + b_3 + \cdots)$$

Do it systematically:

$$a_0 b_0 + (a_0 b_1 + a_1 b_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0) + \cdots$$

Motivation:

$$(\alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots)(\beta_0 + \beta_1 z + \beta_2 z^2 + \cdots)$$
$$= \alpha_0\beta_0 + (\alpha_0\beta_1 + \alpha_1\beta_0) z + (\alpha_0\beta_2 + \alpha_1\beta_1 + \alpha_2\beta_0) z^2 + \cdots$$

Prop Let $a_n, b_n \in \mathbb{C}$. Suppose $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$ are absolutely convergent. Let $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

Then $\sum_{n=0}^{\infty} c_n$ is absolutely convergent and

$$\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} a_n \times \sum_{n=0}^{\infty} b_n .$$

When does a series converge?

Prop (Ratio test). Let $z_n \in \mathbb{C}$.

Suppose $\lim_{n\to\infty} \frac{|z_{n+1}|}{|z_n|} = l$.

If $l < 1$ then $\sum_{n=0}^{\infty} z_n$ is abs. convgt.

$l > 1$ then $\sum_{n=0}^{\infty} z_n$ diverges

$l = 1$ then $\sum_{n=0}^{\infty} z_n$ may be convergent but not absolutely convergent, may diverge, may converge absolutely — can't tell!

Prop (Root test) Let $z_n \in \mathbb{C}$.

Suppose $\lim_{n\to\infty} |z_n|^{1/n} = l$.

If $\quad l < 1$ then $\sum_{n=0}^{\infty} z_n$ is abs. convgt

$\quad l > 1$ then $\sum_{n=0}^{\infty} z_n$ diverges

$\quad l = 1$ who knows?

Example $\sum_{n=0}^{\infty} \frac{i^n}{3^n}$. Here $z_n = \frac{i^n}{3^n}$.

Using the ratio test:

$$\frac{|z_{n+1}|}{|z_n|} = \frac{|i^{n+1}/3^{n+1}|}{|i^n/3^n|} = \left|\frac{i}{3}\right| = \frac{1}{3} \xrightarrow{\text{as } n\to\infty} \frac{1}{3} < 1.$$

By the ratio test, $\sum_{n=0}^{\infty} i^n/3^n$ is abs. convgt.

Using the root test:

$$|z_n|^{1/n} = \left|\frac{i^n}{3^n}\right|^{1/n} = \left(\frac{1}{3^n}\right)^{1/n} = \frac{1}{3} \xrightarrow{\text{as } n\to\infty} \frac{1}{3} < 1.$$

By the root test, $\sum_{n=0}^{\infty} i^n/3^n$ is abs. convgt.

Power series & the radius of convergence

Let $a_n \in \mathbb{C}$, $z_0 \in \mathbb{C}$. A series of the form

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{is called a } \underline{\text{power series}} \text{ in } z.$$

(ie $z_n = a_n(z-z_0)^n$ in the previous notation).

Think of: a_n = coefficients

z_0 = where the power series is centred.

z = variable.

We can change coordinates to make this centred at the origin: $z' = z - z_0$. Hence it is sufficient to consider power series centred at 0.

$$\sum_{n=0}^{\infty} a_n z^n \qquad (\text{here } a_n \in \mathbb{C}) \qquad (*)$$

Q: For which values of z does the power series $(*)$ converge?

$$\text{Let } R := \sup \left\{ r \geq 0 \;\middle|\; \begin{array}{l} \exists z \in \mathbb{C} \text{ s.t } |z| = r \text{ and} \\ \sum_{n=0}^{\infty} a_n z^n \text{ converges} \end{array} \right\}.$$

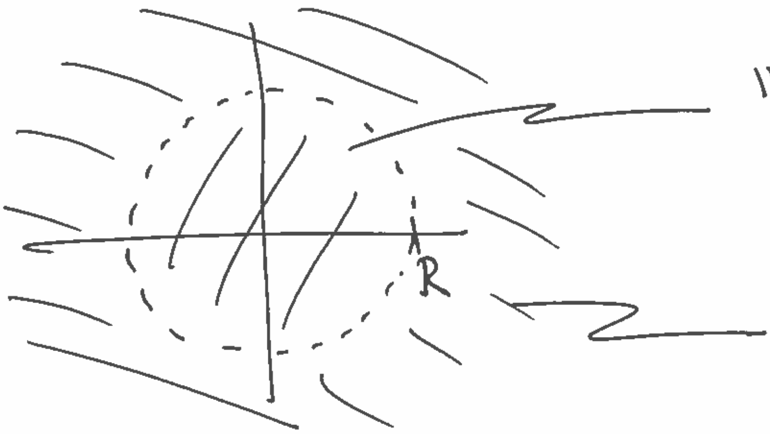
Thm

(1) If $|z| < R$ then $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely.

(2) If $|z| > R$ then $\sum_{n=0}^{\infty} a_n z^n$ diverges.

[If $|z| = R$ then we can't say anything]

Defn We call R the radius of convergence



inside this circle, the power series converges absolutely.

outside this circle, the power series diverges.

If $R = \infty$ then the power series converges (absolutely) for all $z \in \mathbb{C}$.

Prop Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series.

(1) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists then $\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

(2) If $\lim_{n\to\infty} |a_n|^{1/n}$ exists then $\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n}$.

Rmk By convention: $\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$.