Solutions for MATH20142 Complex Analysis solutions, Coursework, March 2020

On completion of this unit successful students will be able to:

ILO1 prove the Cauchy-Riemann Theorem and its converse and use them to decide whether a given function is holomorphic;

ILO2 use power series to define a holomorphic function and calculate its radius of convergence;

ILO3 define the complex integral and use a variety of methods (the Fundamental Theorem of Contour Integration, Cauchy's Theorem, the Generalised Cauchy Theorem and the Cauchy Residue Theorem) to calculate the complex integral of a given function;

ILO4 define and perform computations with elementary holomorphic functions such as sin, cos, sinh, cosh, exp, log, and functions defined by power series;

ILO5 use Taylor's Theorem and Laurent's Theorem to expand a holomorphic function in terms of power series on a disc and Laurent series on an annulus, respectively;

ILO6 identify the location and nature of a singularity of a function and, in the case of poles, calculate the order and the residue;

ILO7 apply techniques from complex analysis to deduce results in other areas of mathematics, including proving the Fundamental Theorem of Algebra and calculating infinite real integrals, trigonometric integrals, and the summation of series.

Question	Learning Outcome	Solution

Q1 — **ILO1 (part (i)), ILO4 (part (ii))**

Assessed at: medium level (i), medium level (ii).

(i), (ii) are similar to exercise sheets

(i) Let $a_n = \frac{(4i)^n}{n}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(4i)^{n+1}}{n+1}\frac{n}{(4i)^n}\right| = \frac{4(n+1)}{n} \to 4 = \frac{1}{R}$$

as $n \to \infty$. Hence the radius of convergence is $1/4$.

[4 marks]

(ii) We have

$$\begin{aligned}\sin z \sinh z &= \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right) \times \left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots\right) \\ &= z^2 + \left(\frac{1}{3!} - \frac{1}{3!}\right) z^4 + \left(\frac{1}{5!} - \frac{1}{3!3!} + \frac{1}{5!}\right) z^6 + \cdots \\ &= z^2 + \frac{-1}{90} z^6 + \cdots .\end{aligned}$$

Hence the coefficient of z^6 is $-1/90$.

[4 marks]

Question	Learning Outcome	Solution

Q2 — **ILO3**

(i) If $z \in A$ then $w(\gamma, z) = 0$.

If $z \in B$ then $w(\gamma, z) = 1$.

Assessed at: low level (i), medium level (ii), high level (iii).

(ii) is bookwork, (i), (iii) are similar to exercise sheets

If $z \in C$ then $w(\gamma, z) = 0$.

If $z \in A$ then $w(\gamma, z) = 2$.

[4 marks]

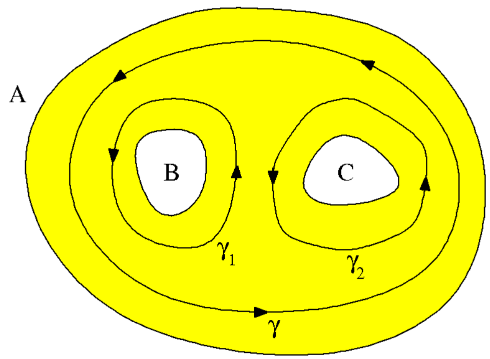
(ii) Let D be a domain and let f be holomorphic on D. Let $\gamma_1, \ldots, \gamma_n$ be closed contours in D. Suppose that

$$w(\gamma_1, z) + \cdots + w(\gamma_n, z) = 0 \quad \text{for all } z \notin D.$$

Then $\int_{\gamma_1} f + \cdots + \int_{\gamma_n} f = 0$.

[2 marks]

(iii) Apply the Generalised Cauchy Theorem to $\gamma_1, \gamma_2, -\gamma$. Let the regions of $\mathbb{C}$ not in D as below



If $z \in A$ then $w(\gamma_1, z) = 0, w(\gamma_2, z) = 0, w(-\gamma, z) = 0$.

If $z \in B$ then $w(\gamma_1, z) = +1, w(\gamma_2, z) = 0, w(-\gamma, z) = -1$.

If $z \in C$ then $w(\gamma_1, z) = -, w(\gamma_2, z) = +1, w(-\gamma, z) = -1$.

In all cases, if $z \notin D$ then $w(\gamma_1, z) + w(\gamma_2, z) + w(-\gamma, z) = 0$. Hence the hypotheses of the Generalised Cauchy Theorem hold.

Hence

$$\int_{\gamma_1} f + \int_{\gamma_2} f + \int_{-\gamma} f = 0.$$

Hence

$$\int_{\gamma} f = \int_{\gamma_1} f + \int_{\gamma_2} f = (1 + 2i) + (3 + 4i) = 4 + 6i.$$

[6 marks]