MATH20101

Three hours

THE UNIVERSITY OF MANCHESTER

REAL AND COMPLEX ANALYSIS

21 January 2019

2.00 – 5.00

Answer **FIVE** questions including at least **TWO** questions in Section A and at least **TWO** questions in Section B. Write your answers for Part A and for Part B in separate booklets. If you answer more than the required number of questions then your best marks, subject to the above constraints, will be used.

University approved electronic calculators may be used.

SECTION A

A1.

(i) Prove, by verifying the ε - δ definition, that

$$\lim_{x\to -1}\left(4x^3+10x^2+4x+1\right)=3.$$

(ii) Prove, by verifying the ε - X definition, that

$$\lim_{x\to +\infty}\frac{4x^2+3x+2}{2x^2+1}=2.$$

(iii) Prove the *Sandwich Rule:* Suppose that f, g and h are three functions for which

$$h(x)\le f(x)\le g(x)$$

for all x in some deleted neighbourhood of $a\in\mathbb{R}$. Assume further that $\lim_{x\to a} h(x)=L$ and $\lim_{x\to a} g(x)=L$. Prove that $\lim_{x\to a} f(x)=L$.

[20 marks]

A2.

(i) State carefully the *Intermediate Value Theorem.*

Prove that

$$e^x=-2x^3+7x$$

has at **least three** solutions in $[-2,2]$.

(ii) Prove that if f is a function continuous on $[a,b]$ then f attains its upper bound. That is, there exists $d\in[a,b]$ such that $f(x)\le f(d)$ for all $x\in[a,b]$.

You may assume that a continuous function on the closed and bounded interval $[a,b]$ is bounded.

(iii) State carefully *Rolle's Theorem.*

Prove that

$$e^x=-2x^3+7x$$

has **exactly** three solutions in $\mathbb{R}$.

[20 marks]

A3.

(i) Given a function f whose first n derivatives exist at $a \in \mathbb{R}$ define the Taylor polynomial $T_{n,a}f(x)$ of degree n at a.

(ii) Prove that if the first $n+1$ derivatives of f exist on an open interval I then

$$\frac{d}{dt}T_{n,t}f(x) = \frac{(x-t)^n}{n!}f^{(n+1)}(t)$$

for all $x, t \in I$.

(iii) Calculate

$$T_{4,0}\left(\left(1-x^2\right)e^x\right).$$

[20 marks]

A4.

(i) Let $f : [a,b] \to \mathbb{R}$ be a bounded function and let $\mathcal{P} = \{x_i : 0 \le i \le n\}$ be a partition of $[a,b]$.

(a) Define the *Upper Sum* $U(\mathcal{P}, f)$ and *Lower Sum* $L(\mathcal{P}, f)$, explaining the meaning of all terms.

(b) Show that if $y \in [a,b]$ and $y \notin \mathcal{P}$ then

$$U(\mathcal{P} \cup \{y\}, f) \le U(\mathcal{P}, f).$$

(ii) Let $f : [1,4] \to \mathbb{R}$, $x \mapsto 1/x^3$ and, for every $n \ge 1$, define the partition

$$\mathcal{Q}_n = \left\{\eta^i : 0 \le i \le n\right\},$$

where $\eta^n = 4$. Prove that

$$L(\mathcal{Q}_n, f) = \frac{15}{16}\frac{1}{\eta\,(\eta+1)}.$$

Find a similar expression for $U(\mathcal{Q}_n, f)$.

Prove, by verifying the definition, that f is integrable over $[1,4]$ and find the value of the integral.

[20 marks]

A5. Recall that a function $f : D \to \mathbb{C}$ is said to be *differentiable at* $z_0 \in \mathbb{C}$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

(i) Let D be a domain and suppose that $f : D \to \mathbb{C}$. State and prove the Cauchy-Riemann Theorem.

(ii) Let $g(z) = \overline{z^2}$. Let $g(x+iy) = u(x,y) + iv(x,y)$ where u, v denote the real and imaginary parts of g, respectively. Show that $u(x,y) = x^2 - y^2$ and that $v(x,y) = -2xy$.

Use the Cauchy-Riemann Theorem to show that g is not differentiable at any point $z_0 \in \mathbb{C} \setminus \{0\}$.

(iii) State, without proof, the Partial Converse to the Cauchy-Riemann Theorem that was discussed in the course.

Is the function g (defined in (ii) above) differentiable at the origin? Justify your answer.

[20 marks]

A6.

(i) Let D be a domain, let $\gamma : [a, b] \to D$ be a path in D, and let $f : D \to \mathbb{C}$ be continuous. Write down the definition of the path integral $\int_\gamma f$.

(ii) Let $\gamma : [a, b] \to D$ be a path which starts at $z_0 = \gamma(a)$ and ends at $z_1 = \gamma(b)$. Recall that the *reversed path*, denoted by $-\gamma$, is the path that starts at z_1, travels 'backwards' along γ, and ends at z_0. A parametrisation of $-\gamma$ is given by $-\gamma(t) = \gamma(a + b - t)$, $a \le t \le b$.

Suppose that $f : D \to \mathbb{C}$ is continuous. Prove from the definition of the path integral that

$$\int_{-\gamma} f = -\int_\gamma f.$$

(iii) State, without proof, the Generalised Cauchy Theorem.

(iv) Let $D = \mathbb{C} \setminus \{-1 + i, 1, -1 - i\}$ and let $\gamma, \gamma_1, \gamma_2, \gamma_3$ be closed contours in D, as illustrated in Figure 1. Suppose that $f : D \to \mathbb{C}$ is holomorphic and

$$\int_{\gamma_1} f = 2 + 2i, \quad \int_{\gamma_2} f = 3 + 3i, \quad \int_{\gamma_3} f = 4 + 4i.$$

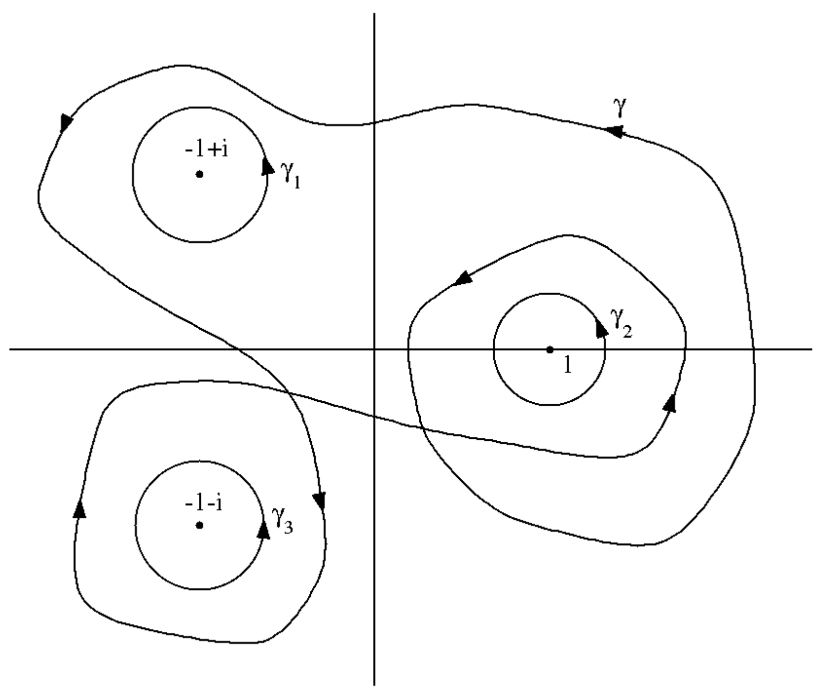

Figure 1: See Question B6(iv).

Write down the winding numbers $w(\gamma, -1 + i), w(\gamma, 1), w(\gamma, -1 - i)$.

Use the Generalised Cauchy Theorem to calculate $\int_\gamma f$.

[20 marks]

A7.

(i) Supppose that f is holomorphic on the disc $\{z \in \mathbb{C} \mid |z - z_0| < R\}$. Let $0 < r < R$. Let C_r denote the circular contour $C_r(t) = z_0 + re^{it}$, $0 \leq t \leq 2\pi$. Taylor's Theorem gives the following expression for the nth derivative of f at z_0:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_r} \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

(you do not need to prove this).

Suppose that there exists $M \geq 0$ such that $|f(z)| \leq M$ for all z such that $|z - z_0| = r$. Use the above expression for $f^{(n)}(z_0)$ and the Estimation Lemma to prove Cauchy's Estimate, namely

$$|f^{(n)}(z_0)| \leq \frac{Mn!}{r^n}.$$

(ii) Liouville's Theorem states the following: Suppose that $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and bounded on $\mathbb{C}$ (that is, there exists $M > 0$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$). Then f is constant.

Deduce Liouville's Theorem from Cauchy's Estimate. (You may assume, without proof, that if $f'(z) = 0$ for all $z \in \mathbb{C}$ then f is constant.)

(iii) Suppose that f is holomorphic on the deleted disc $\{z \in \mathbb{C} \mid 0 < |z - z_0| < R\}$. What does it mean (in terms of Laurent series) to say that z_0 is a pole of order m? What does it mean to say that z_0 is an isolated essential singularity?

The functions below have singularities at 0. In each case, determine the nature of the singularity at 0 and give a reason for your answer. If the singularity is a pole, state the order of the pole.

$$\frac{1}{z^3(1 - z)}, \quad z^3 \sin \frac{1}{z}.$$

(iv) Suppose that $f(z)$ has a pole of order 2 at z_0 and that $g(z)$ has a pole of order 3 at z_0. Let $h(z) = f(z)g(z)$. Prove that h has a pole at z_0 and determine its order.

[20 marks]

A8.

(i) State, without proof, Cauchy's Residue Theorem.

(ii) Suppose that g is meromorphic on $\mathbb{C}$ with poles at 0, $1+i$ and $2+2i$ and residues

$$\mathrm{Res}(g,0) = \frac{2}{i}, \quad \mathrm{Res}(g,1+i) = \frac{3}{i}, \quad \mathrm{Res}(g,2+2i) = \frac{5}{i}.$$

Let γ_1, γ_2 be the closed contours as illustrated in Figure 2. Calculate $\int_{\gamma_1} g$ and $\int_{\gamma_2} g$.

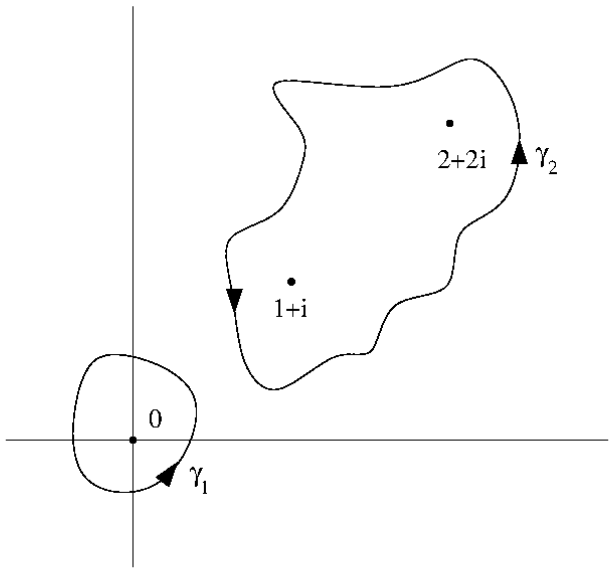

Figure 2: See Question B8(ii).

(iii) Let $f : \mathbb{R} \to \mathbb{R}$. Recall that $\int_{-\infty}^{\infty} f(x)\,dx$ is defined to be

$$\lim_{A\to\infty, B\to\infty} \int_{-A}^{B} f(x)\,dx$$

where the limit is taken in either order, when it exists. The principal value $\mathcal{P}\int_{-\infty}^{\infty} f(x)\,dx$ of the integral is defined to be

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx.$$

State, without proof, a criterion on f which ensures that if the principal value of the integral $\mathcal{P}\int_{-\infty}^{\infty} f(x)\,dx$ exists then so does $\int_{-\infty}^{\infty} f(x)\,dx$ and the two quantities are the same.

(iv) Let

$$h(z) = \frac{1}{z^2+8}.$$

Show that h satisfies the criterion you gave in (iii).

Locate the singularites of h and determine their residues. (You may use any standard results in the course for calculating residues, provided that you state them clearly.)

By integrating around a suitable D-shaped contour, show that

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+8} = \frac{\pi}{2\sqrt{2}}.$$

[20 marks]

END OF EXAMINATION PAPER