Three hours

THE UNIVERSITY OF MANCHESTER

REAL AND COMPLEX ANALYSIS

?? January 2018

??:?? – ??:??

Answer **FIVE** questions including at least **TWO** questions in Section A and at least **TWO** questions in Section B. Write your answers for Part A and for Part B in separate booklets. If you answer more than the required number of questions then your best marks, subject to the above constraints, will be used.

University approved calculators may be used.

SECTION A

A1.

(i) Prove, by verifying the ε - δ definition, that

$$\lim_{x\to 2}\left(x^3-3x^2+6\right)=2.$$

(ii) Prove the *Product Rule for Limits*: Assume that f and g are real valued functions defined on a deleted neighbourhood of $a\in\mathbb{R}$. Further assume that $\lim_{x\to a} f(x)=L$ and $\lim_{x\to a} g(x)=M$. Prove that

$$\lim_{x\to a} f(x)\,g(x)=LM.$$

(You may assume that if $\lim_{x\to a} h\left(x\right)=H$ then $\left|h\left(x\right)\right|<\left|H\right|+1$ in some deleted neighbourhood of a.)

(iii) Using the limit laws evaluate

$$\lim_{x\to -1}\frac{x^3+1}{x^3-4x^2-2x+3}.$$

[20 marks]

A2.

(i) Show, by verifying the definition, that

$$g(x) = \frac{x^2}{1+x}$$

is differentiable on $\mathbb{R} \setminus \{-1\}$ and find its derivative.

(ii) (a) State carefully *Rolle's Theorem.*

(b) State carefully the *Mean Value Theorem.*

(c) Deduce the Mean Value Theorem from Rolle's Theorem.

(iii) Prove that

$$\ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$$

for $x > 0$.

[20 marks]

A3.

(i) State the ε - δ definition that $h : \mathbb{R} \to \mathbb{R}$ is *continuous* at $a \in \mathbb{R}$.

(ii) Assume that g is defined on a deleted neighbourhood of $a \in \mathbb{R}$ and that $\lim_{x\to a} g(x) = L$ exists. Assume that f is defined on a neighbourhood of L and is continuous at L. Prove that

$$\lim_{x\to a} f(g(x)) = f(L)$$

Hint. Consider f first.

(iii) Calculate the Taylor polynomial

$$T_{6,0}\left((1+x)\cos^2 x\right).$$

[20 marks]

A4.

(i) Assume f is a bounded function on the interval $[a, b]$.

(a) Define what is meant by saying that $\mathcal{P}$ *is a partition of* $[a, b]$.

(b) Define the

$$\textit{Upper integral } \overline{\int_a^b} f \quad \text{and} \quad \textit{Lower integral } \underline{\int_a^b} f,$$

not forgetting to define all the terms you use.

(c) Prove that the lower and upper sums satisfy

$$L(\mathcal{Q}, f) \leq U(\mathcal{R}, f)$$

for any partitions $\mathcal{Q}$ and $\mathcal{R}$ of $[a, b]$. (You may assume that $L(\mathcal{P}, f) \leq U(\mathcal{P}, f)$ for any partition $\mathcal{P}$ while

$$L(\mathcal{P}, f) \leq L(\mathcal{D}, f) \quad \text{and} \quad U(\mathcal{D}, f) \leq U(\mathcal{P}, f)$$

whenever $\mathcal{P} \subseteq \mathcal{D}$.)

(d) Deduce that

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

(ii) Let $f : [2, 8] \to \mathbb{R}, x \mapsto 1/x^3$ and, for every $n \geq 1$, define the partition

$$\mathcal{Q}_n = \left\{2\eta^i : 0 \leq i \leq n\right\},$$

where $\eta^n = 4$.

(a) Show that

$$L(\mathcal{Q}_n, f) = \frac{15}{64\eta\,(1+\eta)}.$$

(You may assume that $\sum_{i=1}^n x^i = x\,(1 - x^n)\,/\,(1 - x)$.)

(b) Prove, by verifying the definition, that f is integrable over $[2, 8]$ and find the value of the integral.

(You may assume that $U(\mathcal{Q}_n, f) = 15\eta^2/64\,(1 + \eta)$.)

[20 marks]

SECTION B

B5.

(i) Let $z \in \mathbb{C} \setminus \{0\}$. Suppose that $\exp w = z$ where $w \in \mathbb{C}$. Determine the real and imaginary parts of w in terms involving $|z|$ and $\arg z$.

How is the complex logarithm $\log z$ defined? How is $\operatorname{Log} z$, the principal value of the complex logarithm, defined?

Explain briefly why the principal logarithm is not continuous on $\mathbb{C} \setminus \{0\}$.

(ii) Find two complex numbers $z_1, z_2 \in \mathbb{C} \setminus \{0\}$ for which

$$\operatorname{Log} z_1 z_2 \neq \operatorname{Log} z_1 + \operatorname{Log} z_2.$$

(iii) What does it mean for a function $f : D \to \mathbb{C}$ to be differentiable at a point $z_0 \in D$?

Let D denote the cut plane. By using the fact that $\operatorname{Log} z$ is continuous on the cut plane (you do not need to prove this), prove directly from the definition of differentiability that for all $z \in D$

$$\frac{d}{dz} \operatorname{Log} z = \frac{1}{z}.$$

(iv) Recall that if $b \in \mathbb{C} \setminus \{0\}$ then b^z is defined to be $\exp(z \log b)$ where $\log b$ is any complex logarithm of b. The principal value of b^z is defined to be $\exp(z \operatorname{Log} b)$.

Calculate all the values of $(1+i)^i$. Show that the principal value of $(1+i)^i$ is

$$e^{-\pi/4} \cos(\ln \sqrt{2}) + i e^{-\pi/4} \sin(\ln \sqrt{2}).$$

[20 marks]

B6.

(i) Let D be a domain and let γ be a smooth path in D. Let $f : D \to \mathbb{C}$ be continuous. Write down the definition of $\int_\gamma f$.

(ii) State, and prove, the Fundamental Theorem of Contour Integration.

(iii) Suppose that D is a domain and that γ is a closed contour in D. Suppose that f has an antiderivative F defined on D. What does the Fundamental Theorem of Contour Integration tell you about the value of $\int_\gamma f$?

Let $g(z) = z/(z-1)$. Let C denote the circular contour with centre 1 and radius 2, described once anticlockwise. From the definition of the contour integral you gave in (i), calculate $\int_C g$. What does the Fundamental Theorem of Contour Integration tell you about g?

[20 marks]

B7. Recall that the Laurent expansion of a function f in the annulus $R_1 < |z - z_0| < R_2$ is an expression of the form

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

where $a_n \in \mathbb{C}$ and this double sum converges for z in the annulus $R_1 < |z - z_0| < R_2$.

(i) What does it mean to say that f has a singularity at $z_0 \in \mathbb{C}$? What does it mean to say that z_0 is an isolated singularity?

Briefly explain the role of Laurent's theorem in classifying isolated singularities into removable singularities, poles of order m, and isolated essential singularities.

Suppose that z_0 is a pole of order m. How is $\mathrm{Res}(f, z_0)$, the residue of f at z_0, defined?

(ii) It was proved in the course that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$, is a power series in z with radius of convergence R then $f(z)$ is differentiable on $\{z \in \mathbb{C} \mid |z| < R\}$ and $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ for $|z| < R$.

By considering the sum to infinity of an appropriate geometric progression, use this result to prove that

$$\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + \cdots + nz^{n-1} + \cdots \text{ for } |z| < 1.$$

(iii) Let

$$g(z) = \frac{1}{z(1-z)^2}.$$

Find the Laurent series for g on the annulus $0 < |z| < 1$.

By changing variables to $w = z - 1$, find the Laurent series for g on the annulus $0 < |z - 1| < 1$.

By using these Laurent series, write down the values of $\mathrm{Res}(g, 0)$ and $\mathrm{Res}(g, 1)$. (You may **not** use standard formulæ for the residue at a pole given in the course.)

(iv) Suppose that f is holomorphic on the annulus $D = \{z \in \mathbb{C} \mid 0 < |z - z_0| < R\}$. It was stated in the course that the coefficient b_n of the term $(z - z_0)^{-n}$ in the Laurent series of a function f is given by

$$b_n = \frac{1}{2\pi i} \int_{C_r} f(z)(z - z_0)^{n-1}\, dz$$

where C_r is a circular contour in D with centre z_0, radius $r \in (0, R)$, described once anticlockwise.

Suppose that there exists $M > 0$ such that for all $z \neq 0$ we have $|f(z)| \leq M/|z|$. Use the Estimation Lemma to show that $f : D \to \mathbb{C}$ has either a simple pole or a removable singularity at 0. Give examples to show that both of these possibilities can happen.

[20 marks]

B8.

(i) Let D be a domain and suppose that $f : D \to \mathbb{C}$ is meromorphic and has a simple pole at z_0. Recall the following formula for the residue of f at the simple pole z_0:

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

Let

$$f(z) = \frac{1}{3z^2 + 10iz - 3}.$$

Show that f has simple poles at $z = -i/3, -3i$. Show that $\operatorname{Res}(f, -i/3) = 1/8i$ and calculate $\operatorname{Res}(f, -3i)$.

(ii) State, without proof, Cauchy's Residue Theorem.

(iii) Show that

$$\int_0^{2\pi} \frac{1}{5 + 3\sin t}\, dt = 2 \int_{C_1} f(z)\, dz$$

where $f(z) = 1/(3z^2 + 10iz - 3)$ and C_1 denotes the circle in $\mathbb{C}$, centred at the origin and with radius 1, described once anticlockwise.

Using Cauchy's Residue Theorem, calculate

$$\int_0^{2\pi} \frac{1}{5 + 3\sin t}\, dt.$$

Explain why Cauchy's Residue Theorem does not allow you to calculate

$$\int_0^{2\pi} \frac{1}{3 + 3\sin t}\, dt.$$

[20 marks]

END OF EXAMINATION PAPER