## Three hours

# THE UNIVERSITY OF MANCHESTER 

## REAL AND COMPLEX ANALYSIS

?? January 2017
??:?? - ??:??

Answer FIVE questions including at least TWO questions in Section A and at least TWO questions in Section B. Write your answers for Part A and for Part B in separate booklets. If you answer more than the required number of questions then your best marks, subject to the above constraints, will be used.

Electronic calculators may be used, provided that they cannot store text/transmit or receive information/display graphics.

## SECTION A

A1.
(i) Prove, by verifying the $\varepsilon-\delta$ definition, that

$$
\lim _{x \rightarrow 3-} \frac{x^{2}-9}{|x-3|}=-6
$$

(ii) Prove, by verifying the $X-\delta$ definition, that

$$
\lim _{x \rightarrow 2} \frac{x}{(x-2)^{2}}=+\infty
$$

(iii) Assume that $g: A \rightarrow \mathbb{R}$ is defined on a deleted neighbourhood of $a \in \mathbb{R}$ and $\lim _{x \rightarrow a} g(x)=L$ with $L \neq 0$.

Prove, by verifying the definition, that

$$
\lim _{x \rightarrow a} \frac{1}{g(x)}=\frac{1}{L}
$$

(You may assume that $|g(x)|>|L| / 2$ in some deleted neighbourhood of $a$.)

A2.
(i) (a) State carefully the Intermediate Value Theorem.

Prove that

$$
\cos x+2 \cos \left(\frac{x}{2}\right)=\sin x+2 \sin \left(\frac{x}{2}\right)
$$

has a solution in $[0, \pi / 2]$.
(b) State carefully Rolle's Theorem.

Prove that

$$
\cos x+2 \cos \left(\frac{x}{2}\right)=\sin x+2 \sin \left(\frac{x}{2}\right)
$$

has exactly one solution in $[0, \pi / 2]$.
(ii) Assume that $f$ is differentiable on $(a, b)$ with a local maximum at $c \in(a, b)$. Prove that $f^{\prime}(c)=0$.

A3.
(i) Suppose that $f$ and $g$ are two functions differentiable at $a \in \mathbb{R}$. Prove the Product Rule for Differentiation, namely that

$$
(f g)^{\prime}(a)=f(a) g^{\prime}(a)+f^{\prime}(a) g(a)
$$

using the Rules for Limits.
(ii) Give an example of functions $f$ and $g$ not differentiable at a point $a \in \mathbb{R}$ but for which the product $f g$ is differentiable at $a$.
(iii) Calculate the Taylor Polynomial

$$
T_{6,0}\left(e^{x} \cos x\right)
$$

[20 marks]
A4.
(i) Let $f:[a, b] \rightarrow \mathbb{R}$ be a bounded function and let $\mathcal{P}$ be a partition of $[a, b]$,

$$
a=x_{0}<x_{1}<x_{2}<\ldots<x_{n-1}<x_{n}=b .
$$

Define the Upper Sum $U(\mathcal{P}, f)$ and Lower $\operatorname{Sum} L(\mathcal{P}, f)$, explaining fully the meaning of all terms.
(ii) Let $h:[0,1] \rightarrow \mathbb{R}$ be given by

$$
h(x)= \begin{cases}1 & \text { if } x \text { is rational } \\ 0 & \text { if } x \text { is irrational. }\end{cases}
$$

Prove that $h$ is not Riemann integrable over $[0,1]$.
(iii) Let $f:[1,4] \rightarrow \mathbb{R}, x \mapsto x^{2}-1$ and, for every $n \geq 1$, define

$$
\mathcal{P}_{n}=\left\{1+\frac{3 i}{n}: 0 \leq i \leq n\right\}
$$

(a) Prove that

$$
U\left(\mathcal{P}_{n}, f\right)=\frac{9(n+1)(4 n+1)}{2 n^{2}}
$$

(You may assume that $\sum_{i=1}^{n} i=n(n+1) / 2$ and $\sum_{i=1}^{n} i^{2}=n(n+1)(2 n+1) / 6$.)
(b) Find a similar result for $L\left(\mathcal{P}_{n}, f\right)$.
(c) Prove, by verifying the definition, that $f$ is integrable over $[1,4]$ and find the value of the integral.

## SECTION B

## B5.

(i) Let $g: \mathbb{R}^{2} \rightarrow \mathbb{R}$ be a function. Write down the definition of the partial derivative

$$
\frac{\partial g}{\partial x}\left(x_{0}, y_{0}\right)
$$

of $g$ at the point $\left(x_{0}, y_{0}\right) \in \mathbb{R}^{2}$, assuming it exists.
Let $D \subset \mathbb{C}$ be a domain and let $f: D \rightarrow \mathbb{C}$. Let $z_{0} \in D$. What does it mean to say that $f$ is differentiable at $z_{0}$ ? What does it mean to say that $f$ is holomorphic on $D$ ?
(ii) Suppose that $D$ is a domain and that $f: D \rightarrow \mathbb{C}$ is holomorphic. Let $z=x+i y$ and write $f(z)=f(x+i y)=u(x, y)+i v(x, y)$ where $u, v$ are real-valued. Let $z_{0}=x_{0}+i y_{0} \in D$. Prove from the definition you gave in (i) that

$$
f^{\prime}\left(z_{0}\right)=\frac{\partial v}{\partial y}\left(x_{0}, y_{0}\right)-i \frac{\partial u}{\partial y}\left(x_{0}, y_{0}\right)
$$

(iii) Recall that the Cauchy-Riemann Theorem states that: if $f: D \rightarrow \mathbb{C}, f(x+i y)=u(x, y)+$ $i v(x, y)$, is differentiable at $z_{0}=x_{0}+i y_{0}$ then the partial derivatives of $u$ and $v$ exist at ( $x_{0}, y_{0}$ ) and the Cauchy-Riemann equations hold.
State the (partial) Converse to the Cauchy-Riemann Theorem discussed in the course, namely: state, without proof, sufficient conditions on the partial derivatives of $u$ and $v$ that ensure that $f(x+i y)=u(x, y)+i v(x, y)$ is differentiable at $z_{0} \in D$.
(iv) Suppose that $f(x+i y)=u(x, y)+i v(x, y)$ is holomorphic on $\mathbb{C}$ and has the property that the imaginary part $v(x, y)$ of $f$ is constant. Prove that $f$ is equal to a constant function.
(v) Let $f(x+i y)=\left(x y^{2}\right)^{1 / 3}$. Show, using the definition of the partial derivative that you gave in (i), that the Cauchy-Riemann equations are satisfied at the origin. Show that $f$ is not differentiable at the origin. Why does this not contradict the Cauchy-Riemann Theorem?
[20 marks]

B6.
(i) Recall that for $z \in \mathbb{C}$ we define

$$
\sin z=\frac{e^{i z}-e^{-i z}}{2 i}
$$

Show from this definition that if $z=x+i y$ then

$$
\sin z=\sin x \cosh y+i \cos x \sinh y
$$

(ii) Show that the solutions $z \in \mathbb{C}$ to $\sin z=0$ are precisely $z=k \pi$ where $k \in \mathbb{Z}$. (You may assume without proof properties of $\sin x, \cos x, \sinh x, \cosh x$ when $x$ is real.)
(iii) Recall that $p \in \mathbb{C}$ is said to be a period of $f: \mathbb{C} \rightarrow \mathbb{C}$ if $f(z+p)=f(z)$ for all $z \in \mathbb{C}$.

Show that the periods of $\sin z$ are precisely $p=2 k \pi$ for $k \in \mathbb{Z}$. (You may use, without proof, the fact that $\sin (z+w)=\sin z \cos w+\cos z \sin w$ for all $z, w \in \mathbb{C}$.)
Show that the periods of $\exp z$ are precisely $p=2 k \pi i, k \in \mathbb{Z}$, stating clearly any standard properties of the exponential function that you use.
(iv) Find all complex solutions $z \in \mathbb{C}$ to the equation $e^{z}=e^{i z}$.

## B7.

(i) Consider the closed path $\gamma$ illustrated in Figure 1 below. Calculate (by eye) the winding number of $\gamma$ around any point in each of the regions $A, B, C, D$.


Figure 1: See question B7(i).
(ii) Let $D \subset \mathbb{C}$ be a domain and let $f: D \rightarrow \mathbb{C}$ be continuous. Let $\gamma:[a, b] \rightarrow D$ be a smooth path. Write down the definition of $\int_{\gamma} f$.
(iii) Recall that in the course we derived the following expression for $w\left(\gamma, z_{0}\right)$, the winding number of a closed path $\gamma$ around a point $z_{0}$ :

$$
\begin{equation*}
w\left(\gamma, z_{0}\right)=\frac{1}{2 \pi i} \int_{\gamma} \frac{d z}{z-z_{0}} . \tag{1}
\end{equation*}
$$

Let $\gamma_{1}$ denote the circular path with centre $2+3 i$ and radius 2 , described once anticlockwise (as illustrated on the following page in Figure 2). Write down a parametrisation for $\gamma_{1}$. By using the formula in (1), explicitly calculate $w\left(\gamma_{1}, 2+3 i\right)$.
(iv) State, without proof, the Generalised Cauchy Theorem.

Let $D$ denote the domain $D=\mathbb{C} \backslash\{2+3 i\}$ (see Figure 2). Suppose that $f: D \rightarrow \mathbb{C}$ is holomorphic and $\int_{\gamma_{1}} f=1+i$. Consider the closed contour $\gamma_{2}$ illustrated in Figure 2. By using the Generalised Cauchy Theorem applied to $\gamma_{1}, \gamma_{1}$ and $-\gamma_{2}$, calculate $\int_{\gamma_{2}} f$. (You may calculate winding numbers of $\gamma_{2}$ by eye.)


Figure 2: See question B7(iii), (iv).

## B8.

(i) Recall that $\int_{-\infty}^{\infty} f(x) d x$ is defined to be

$$
\begin{equation*}
\lim _{A, B \rightarrow \infty} \int_{-A}^{B} f(x) d x \tag{2}
\end{equation*}
$$

where the limit is taken in either order. Also recall that the principle value of $\int_{-\infty}^{\infty} f(x) d x$ is defined to be

$$
\begin{equation*}
\lim _{R \rightarrow \infty} \int_{-R}^{R} f(x) d x \tag{3}
\end{equation*}
$$

Give an example of a function $f: \mathbb{R} \rightarrow \mathbb{R}$ for which (3) exists but (2) does not.
State, without proof, a condition on $f$ that ensures that both (2) and (3) exist and the two limits are the same.
(ii) Let

$$
f(z)=\frac{z^{2}}{\left(z^{2}+4\right)\left(z^{2}+9\right)}
$$

Let $R>0$. Let $\Gamma_{R}=[-R, R]+S_{R}$ denote the $D$-shaped contour discussed in the course. Here $[-R, R]$ denotes the straightline path from $-R$ to $R$ along the real axis and $S_{R}$ denotes the semi-circular path from $R$ to $-R$.
Show that $f(z)$ has simple poles at $\pm 2 i$ and $\pm 3 i$. Calculate $\operatorname{Res}(f, 2 i)$ and $\operatorname{Res}(f, 3 i)$. (You may use, without proof, any standard results from the course on calculating the residue at a simple pole.)
Use Cauchy's Residue Theorem to show that if $R>3$ then

$$
\int_{\Gamma_{R}} f(z) d z=\frac{\pi}{5}
$$

Use the Estimation Lemma to show that $\lim _{R \rightarrow \infty} \int_{S_{R}} f(z) d z=0$.
Hence calculate

$$
\int_{-\infty}^{\infty} \frac{x^{2}}{\left(x^{2}+4\right)\left(x^{2}+9\right)} d x
$$

(iii) Write down (without using Cauchy's Residue Theorem) the value of

$$
\int_{-\infty}^{\infty} \frac{x}{\left(x^{2}+4\right)\left(x^{2}+9\right)} d x
$$

briefly justifying your answer.

