Question	Learning Outcome	Solution
B5	Prove the Cauchy-Riemann Theorem and its converse and use them to decide whether a given function is holomorphic. Assessed at: low level (i), medium level (ii), high level (iii). (i) is bookwork, (ii) is similar to example sheets, (iii) is similar to example sheets	

(i) The Cauchy-Riemann Theorem: Let $f : D \to \mathbb{C}$ be differenitable at $z_0 = x_0 + iy_0 \in D$. Write $f(x+iy) = u(x,y) + iv(x,y), u, v : D \to \mathbb{R}$. Then $\partial u/\partial x, \partial u/\partial y, \partial v/\partial x, \partial v/\partial y$ exist at (x_0, y_0) and $\partial u/\partial x(x_0, y_0) = \partial v/\partial y(x_0, y_0)$, $\partial u/\partial y(x_0, y_0) = -\partial v/\partial x(x_0, y_0)$.

Proof. Let $h \in \mathbb{R}$ and consider $z = z_0 + h = x_0 + h + iy_0$. Then $z \to z_0$ as $h \to 0$. Hence

$$\begin{aligned} f'(z_0) &= \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} \\ &= \lim_{h\to 0} \frac{u(x_0+h, y_0) - u(x_0, y_0)}{h} + i\frac{v(x_0+h, y_0) - v(x_0, y_0)}{h} \\ &= \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0). \end{aligned}$$

Let $k \in \mathbb{R}$ and consider $z = z_0 + ik = x_0 + i(y_0 + k)$. Then $z \to z_0$ as $k \to 0$. Hence

$$\begin{aligned} f'(z_0) &= \lim_{k\to 0} \frac{f(z_0+ik) - f(z_0)}{ik} \\ &= \lim_{k\to 0} \frac{u(x_0, y_0+k) - u(x_0, y_0)}{ik} + i\frac{v(x_0, y_0+k) - v(x_0, y_0)}{k} \\ &= \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0). \end{aligned}$$

Hence the partial derivatives all exist. Comparing the real and imaginary parts gives the Cauchy-Riemann equations.

> **Feedback.** Almost everybody knew the Cauchy-Riemann equations. A significant number of people didn't attempt the proof.

[8 marks]

(ii) Let $g(x+iy) = \overline{(x+iy)^2}$. Then

$$g(x+iy) = \overline{(x+iy)(x+iy)} = \overline{x^2 - y^2 + 2ixy} = x^2 - y^2 - 2ixy.$$

Hence $u(x,y) = x^2 - y^2$, $v(x,y) = -2xy$. Also

$$\frac{\partial u}{\partial x} = 2x, \ \frac{\partial v}{\partial y} = -2x$$

$$\frac{\partial u}{\partial y} = -2y, \ -\frac{\partial v}{\partial x} = 2y.$$

Hence if $(x_0, y_0) \neq (0,0)$ then then Cauchy-Riemann equations do not hold at (x_0, y_0). Hence g is not differentiable at $z_0 \neq 0$.

Feedback. Almost everybody who attempted this got this correct.

[6 marks]

(iii) Suppose that $f : D \to \mathbb{C}$, $f(x + iy) = u(x, y) + iv(x, y)$. Suppose $z_0 \in D$ and that

(i) the partial derivatives $\partial u/\partial x, \partial u/\partial y, \partial v/\partial x, \partial v/\partial y$ exist at (x_0, y_0),

(ii) the Cauchy-Riemann equations hold at (x_0, y_0),

(iii) the partial derivatives $\partial u/\partial x, \partial u/\partial y, \partial v/\partial x, \partial v/\partial y$ are continuous at (x_0, y_0).

Then f is differentiable at z_0.

From (ii) we know that the partial derivatives exist at $(0, 0)$. They are polynomial functions and so are continuous. Moreover,

$$\frac{\partial u}{\partial x}(0,0) = 0 = \frac{\partial v}{\partial y}(0,0), \quad \frac{\partial u}{\partial y}(0,0) = 0 = -\frac{\partial v}{\partial x}(0,0).$$

Hence by the partial converse to the Cauchy-Riemann Theorem, g is differentiable at the origin.

Feedback. There were many mis-statements of the Partial Converse to the Cauchy-Riemann Theorem; usually the hypothesis that the partial derivatives are assumed to be continuous was omitted. The second part of this question ('is g differentiable at the origin?') was not well answered. You just need to check that the g satisfies the hypotheses of the Partial Converse to the Cauchy-Riemann Theorem. Many people tried to do this from first principles (specifically: attempting to show that $\lim_{z \to z_0} g(z) - g(z_0)/(z - z_0)$ exists at $z_0 = 0$). This approach does work but involves slightly more work and care with taking limits.

[6 marks]

Question	Learning Outcome	Solution
B6	Define the complex integral and use a variety of methods (the Fundamental Theorem of Contour Integration, Cauchy's Theorem, the Generalised Cauchy Theorem and the Cauchy Residue Theorem) to calculate the complex integral of a given function. Assessed at: low level (i), (iii), medium level (ii), high level (iv). (i) is bookwork, (ii) is from the exercises, (iii) is bookwork, (iv) is unseen/similar to exercises.	(i) $$\int_\gamma f := \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$ **Feedback.** This is a standard definition from the course, and one which we've used many times in the course. [2 marks]

(ii) By the chain rule, $(-\gamma)'(t) = -(\gamma'(a+b-t))$. Hence

$$\begin{aligned}
\int_{-\gamma} f &= \int_a^b f(-\gamma(t))(-\gamma)'(t)\,dt \\
&= -\int_a^b f(\gamma(a+b-t))\gamma'(a+b-t)\,dt \\
&= \int_b^a f(\gamma(u))\gamma'(u)\,du \text{ (substituting } u = a+b-t) \\
&= -\int_a^b f(\gamma(t))\gamma'(t)\,dt \\
&= -\int_\gamma f.
\end{aligned}$$

Feedback. This proved to be tricky, given that in the proof you need to change the sign three times to get the result. Most students didn't get it right. Note that it was one of the exercises in the course.

[6 marks]

(iii) The Generalised Cauchy Theorem: Let D be a domain and let $f : D \to \mathbb{C}$ be holomorphic. Let $\gamma_1, \ldots, \gamma_n$ be closed contours in D such that

$$w(\gamma_1, z) + \cdots + w(\gamma_n, z) = 0 \text{ for all } z \notin D.$$

Then

$$\int_{\gamma_1} f + \cdots + \int_{\gamma_n} f = 0.$$

Feedback. Most of you got this right, although many people stated that f only needs to be continuous; this isn't sufficient, one needs f to be holomorphic for the GCT to hold.

[2 marks]

(iv)

$$w(\gamma, -1+i) = 1, \quad w(\gamma, 1) = 2, \quad w(\gamma, -1-i) = -1.$$

Note that $w(\gamma_1, -1+i) = w(\gamma_2, 1) = w(\gamma_3, -1-i) = 1$ but $w(\gamma_j, z) = 0$ for other points $z \notin D$.

Apply the Generalised Cauchy Theorem to: $\gamma, -\gamma_1, -\gamma_2, -\gamma_2, \gamma_3$. Then

$$w(\gamma, z) + w(-\gamma_1, z) + w(-\gamma_2, z) + w(-\gamma_2, z) + w(\gamma_3, z) = \begin{cases} 1 - 1 + 0 + 0 + 0 = 0 & z = -1+i \\ 2 - 0 - 1 - 1 + 0 = 0 & z = 1 \\ -1 + 0 + 0 + 0 + 1 = 0 & z = -1-i. \end{cases}$$

Hence

$$\int_\gamma f + \int_{-\gamma_1} f + \int_{-\gamma_2} f + \int_{-\gamma_2} f + \int_{\gamma_3} f = 0.$$

Hence

$$\int_\gamma f = \int_{\gamma_1} f + 2\int_{\gamma_2} f - \int_{\gamma_3} f = (2+2i) + 2(3+3i) - (4+4i) = 4+4i.$$

Feedback. Most people demonstrated that they knew the method and idea, but there were many slips in the details.

[10 marks]

Question	Learning Outcome	Solution
B7	Use Taylor's Theorem and Laurent's Theorem to expand a holomorphic function in terms of power series on a disc and Laurent series on an annulus, respectively. Assessed at medium level (i), (ii). Identify the location and nature of a singularity of a function and, in the case of poles, calculate the order and the residue. Assessed at medium level (iii), high level (iv). (i) is bookwork, (ii) is bookwork, (iii) is similar to example sheets, (iv) is similar to example sheets.	(i) Note that if $z \in C_r$. then $\|f(z)/(z-z_0)^{n+1}\| \leq M/r^{n+1}$. By the Estimation Lemma and Taylor's Theorem, $$\begin{aligned} \|f^{(n)}(z_0)\| &\leq \frac{n!}{2\pi} \times (\text{bound on integrand}) \times \text{length}(C_r) \\ &\leq \frac{n!}{2\pi} \times \frac{M}{r^{n+1}} \times 2\pi r \\ &= \frac{Mn!}{r^n}. \end{aligned}$$ **Feedback.** This is a standard piece of bookwork and an easy application of the Estimation Lemma. Several students tried to apply the Estimation Lemma to parts of the integrand separately, which led to mistakes. [4 marks] (ii) Choose $M > 0$ such that $\|f(z)\| \leq M$ for all $z \in \mathbb{C}$. Let $z_0 \in \mathbb{C}$. Since f is holomorphic on $\mathbb{C}$ it is holomorphic on $\{z \in \mathbb{C} \mid \|z - z_0\| < R\}$ for any $R \geq 0$. Let $0 \leq r \leq R$. By (i) we have $\|f'(z_0)\| \leq M/r$. By taking R as large as we please, we can take r as large as we please. Hence by letting $r \to \infty$, we have $\|f'(z_0)\| = 0$. As z_0 is arbitrary, we have $f'(z) = 0$ for all $z \in \mathbb{C}$. Hence f is constant. **Feedback.** This is a proof that we did in the lectures. Most students who attempted it got it right, albeit some with significant gaps in the middle. [6 marks] (iii) By Laurent's Theorem, f has a Laurent series on $\{z \in \mathbb{C} \mid 0 < \|z - z_0\| < r\}$ of the form $$\sum_{n=1}^{\infty} b_n (z-z_0)^{-n} + \sum_{n=0} a_n (z-z_0)^n.$$ z_0 is a pole of order m if $b_n = 0$ for all $n > m$ and $b_m \neq 0$. z_0 is an isolated singularity if $b_n \neq 0$ for infinitely many n.

$$
\begin{aligned}
\frac{1}{z^3(1-z)} &= \frac{1}{z^3}(1+z+z^2+\cdots+z^n+\cdots) \text{ if } |z|<1 \\
&= \frac{1}{z^3}+\frac{1}{z^2}+\frac{1}{z}+1+z+\cdots+z^n+\cdots.
\end{aligned}
$$

Hence there is a pole of order 3 at the origin.

$$z^3 \sin 1/z = z^3\left[\frac{1}{z}-\frac{1}{3!z^3}+\frac{1}{5!z^5}-\cdots+\frac{(-1)^n}{(2n+1)!z^{2n+1}}+\cdots\right].$$

There are infinitely many terms in the principal part, so 0 is an isolated essential singularity.

Feedback. The definitions of pole of order m and isolated essential singularity have proved to be tricky. Many students were too verbose instead of writing a clear formula. The rest of the question went reasonably well. Remember that, in mathematics, if you are asked for a definition then you need to state the precise and accurate definition, and not discuss what it means in general, more hand-wavey, terms.

[6 marks]

(iv) As f has a pole of order 2 at z_0 it has a Laurent series

$$f(z)=\frac{b_2}{(z-z_0)^2}+\frac{b_1}{(z-z_0)}+\sum_{n=0}^{\infty}a_n(z-z_0)^n,\ b_2\neq 0.$$

Hence $F(z):=(z-z_0)^2 f(z)$ is holomorphic on a neighbourhood of z_0 and $F(z_0)\neq 0$.
Similarly, $G(z):=(z-z_0)^3 g(z)$ is holomorphic on a neighbourhood of z_0 and $G(z_0)\neq 0$.
Hence $(z-z_0)^2 f(z)(z-z_0)^3 g(z)=F(z)G(z)=:H(z)$ is holomorphic on a neighbourhood of z_0 and $H(z_0)\neq 0$.
Hence

$$(z-z_0)^5 f(z)g(z)=\sum_{n=0}^{\infty}c_n(z-z_0)^n,\ c_0\neq 0$$

$$h(z)=\frac{c_0}{(z-z_0)^5}+\cdots.$$

So $h(z)$ has a pole of order 5 at z_0.

Feedback. I was pleased at how well those who attempted the question did on it. Some of the explanations were a bit too brief and it was hard to follow exactly what you were doing.

[4 marks]

Question	**Learning Outcome**	**Solution**

B8

Define the complex integral and use a variety of methods (the Fundamental Theorem of Contour Integration, Cauchy's Theorem, the Generalised Cauchy Theorem and the Cauchy Residue Theorem) to calculate the complex integral of a given function;

Assessed at low level (i), medium level (ii).

Identify the location and nature of a singularity of a function and, in the case of poles, calculate the order and the residue.

Assessed at medium level (iv).

Apply techniques from complex analysis to deduce results in other areas of mathematics, including proving the Fundamental Theorem of Algebra and calculating infinite real integrals, trigonometric integrals, and the summation of series.

Assessed at low level (iii), medium level (iv).

(i) is bookwork, (ii) is unseen but similar to example sheets, (iii) is bookwork, (iv) is similar to example sheets.

(i) Cauchy's Residue Theorem: Let D be a domain containing a simple closed loop γ and the points inside γ. Suppose that f is meromorphic on D with finitely many poles at $z_1, \ldots, z_n$ inside γ. Then

$$\int_\gamma f = 2\pi i \sum_{j=1}^{n} \mathrm{Res}(f, z_j).$$

Feedback. This is an important result in the course. Many of you just wrote down that $\int_\gamma f = 2\pi i \sum_{j=1}^{n} \mathrm{Res}(f, z_j)$ without stating any hypotheses of f or γ, or did not explain what the z_j are. A theorem has hypotheses and conclusions—you need to state both.

[2 marks]

(ii)

$$\int_{\gamma_1} g = 2\pi i \mathrm{Res}(g, 0) = 2\pi i \times \frac{2}{i} = 4\pi.$$

$$\int_{\gamma_1} g = 2\pi i(\mathrm{Res}(g, 1+i) + \mathrm{Res}(g, 2+2i)) = 2\pi i \times \left(\frac{3}{i} + \frac{5}{i}\right) = 16\pi.$$

Feedback. Most people answered this well.

[4 marks]

(iii) There exists $C > 0$, $K > 0$, $r > 1$ such that $|f(x)| \leq C/|x|^r$ if $|x| \geq K$.

Feedback. Most people gave the correct criterion.

[2 marks]

(iv) Let $h(x) = 1/(x^2+8)$. Then $|h(x)| \leq 1/x^2$; so h satisfies the criterion in (iii).

$h(z)$ is differentiable except when the denominator vanishes. Note that z^2+8 has simple zeros at $\pm 2i\sqrt{2}$. Hence h has simple poles at $\pm 2i\sqrt{2}$.

If z_0 is a simple pole of h then $\mathrm{Res}(h, z_0) = \lim_{z \to z_0}(z - z_0)h(z)$.

Hence

$$\mathrm{Res}(h, 2i\sqrt{2}) = \lim_{z \to 2i\sqrt{2}} (z - 2i\sqrt{2}) \times \frac{1}{(z-2i\sqrt{2})(z+2i\sqrt{2})} = \frac{1}{4i\sqrt{2}}.$$

$$\mathrm{Res}(h, -2i\sqrt{2}) = \lim_{z \to -2i\sqrt{2}} (z + 2i\sqrt{2}) \times \frac{1}{(z-2i\sqrt{2})(z+2i\sqrt{2})} = \frac{1}{-4i\sqrt{2}}.$$

Let $\Gamma_R = [-R, R] + S_R$ denote the D-shaped contour where S_R is the positive semi-circle with centre 0 and radius R.

If $R > 2\sqrt{2}$ then $2i\sqrt{2}$ is inside Γ_R. By the residue theorem,

$$\int_{\Gamma_R} h = 2\pi i \mathrm{Res}(h, 2i\sqrt{2}) = 2\pi i \times \frac{1}{4i\sqrt{2}} = \frac{\pi}{2\sqrt{2}}.$$

If $z \in S_R$ then $|z^2 + 8| \geq |z|^2 - 8 = R^2 - 8$. Hence

$$\left| \int_{S_R} h \right| \leq \frac{\pi R}{R^2 - 8} \to 0$$

as $R \to \infty$.

Hence

$$\lim_{R\to\infty} \int_{-R}^{R} h(x)\,dx + \lim_{R\to\infty} \int_{S_R} h(z)\,dz = \lim_{R\to\infty} \int_{\Gamma_R} h(z)\,dz = \frac{\pi}{2\sqrt{2}}$$

so that $\int_{-\infty}^{\infty} h(x)\,dx = \pi/2\sqrt{2}$.

Feedback. Most students answered this well. Common mistakes included:

- Not calculating $\mathrm{Res}(h, -2i\sqrt{2})$. Whilst calculating $\int_{-\infty}^{\infty} h(x)\,dx = \pi/2\sqrt{2}$ doesn't need you to find $\mathrm{Res}(h, -2i\sqrt{2})$, the question does ask you to work it out.
- Not showing that $\int_{S_R} h(z)\,dz \to 0$ as $R \to \infty$ or not using the reverse triangle inequality correctly. Many of you wrote that $|z^2 + 8| \geq |z^2| = R^2$ if $z \in S_R$. This isn't correct (example: take $z = 8i \in S_8$, then $|z^2 + 8| = |-64 + 8| = 56 < 64$). You need to use the reverse triangle inequality to say that $|z^2 + 8| \geq |z|^2 - 8 = R^2 - 8$ if $z \in S_R$.

[12 marks]

Final comments. Overall, the complex analysis part of MATH20101 went well and I was pleased by the overall performance of the class. I hope you enjoyed the course and felt that you have learned something from it.

Charles Walkden
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