Feedback on the Complex Analysis part of MATH20101, January 2018

B5 (i) Suppose $e^w = z$. We want to find the real and imaginary parts of w. Write $w = x + iy$ so that $z = e^{x+iy} = e^x e^{iy}$. Take the modulus of both sides to obtain $e^x = |z|$, so that $x = \ln |z|$. Similarly take the argument of both sides to obtain $y = \arg z$.

The complex logarithm is defined to be $\log z = \ln |z| + i \arg z$ and the principal logarithm is defined to be $\operatorname{Log} z = \ln |z| + i \operatorname{Arg} z$ (many of you wrote $\log z = \ln |z| + \arg z$). Here $\operatorname{Arg} z$ is the principal value of the argument of z.

$\operatorname{Log} z$ is not continuous on $\mathbb{C} \setminus \{0\}$ as the principal value of the argument (which is the value of the argument of z that lies between $(-\pi, \pi]$) 'jumps' discontinuously from near $-\pi$ to near π as z moves across the negative real axis. (The reason that $\operatorname{Log} z$ is not continuous on $\mathbb{C} \setminus \{0\}$ is not because it is not differentiable at the origin ($f(z) = 1/z$ is an example of a continuous function on $\mathbb{C} \setminus \{0\}$ that is not differentiable at the origin). Not is it because $\operatorname{Log} z$ is only differentiable on the cut-plane; whilst this statement is true, that does not prevent $\operatorname{Log} z$ being defined on the (larger) set $\mathbb{C} \setminus \{0\}$.)

(ii) The trick here is to understand how the principal value of the argument works. Remember that $\operatorname{Arg} z$ is the unique value of the argument that lies in $(-\pi, \pi]$. If you multiply two complex numbers together, then their arguments add (but we may have to add/subtract multiples of 2π to obtain the principal value of the argument of the product).

Take, for example, $z_1 = z_2 = -i$. Then $\operatorname{Arg} z_1 = \operatorname{Arg} z_2 = -\pi/2$. However, $\operatorname{Arg} z_1 z_2 = \operatorname{Arg} -1 = \pi$. Hence $\operatorname{Log} z_1 = \operatorname{Log} z_2 = \ln |-i| - i\pi/2 = -i\pi/2$ (as $\ln |-i| = \ln 1 = 0$), but $\operatorname{Log} z_1 z_2 = \ln |-1| + i\pi = i\pi$.

Note that if you write down two values of z_1, z_2 then you do need to check that they satisfy the required property.

(iii) f is differentiable at $z_0 \in D$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

Proving that $\operatorname{Log} z$ is holomorphic on the cut-plane is a proof from the course (see Proposition 3.4.4 in the notes). The trick is to rewrite $(\operatorname{Log} z - \operatorname{Log} z_0)/(z - z_0)$ in terms of the exponential function, which we already know how to differentiate. Trying to prove it by writing, for example, $(\operatorname{Log} z - \operatorname{Log} z_0)/(z - z_0) = \operatorname{Log}(z/z_0)/(z - z_0)$ won't work.

(iv) First note that $|1+i| = \sqrt{2}$ and $\arg(1+i) = \pi/4 + 2n\pi$, $n \in \mathbb{Z}$. Hence

$$\begin{aligned} \log(1+i) &= \ln\sqrt{2} + i\left(\frac{\pi}{4} + 2n\pi\right) \\ \mathrm{Log}(1+i) &= \ln\sqrt{2} + i\pi/4. \end{aligned}$$

Hence

$$(1+i)^i = \exp(i\log(1+i)) = \exp\left(-\left(\frac{\pi}{4} + 2n\pi\right) + i\ln\sqrt{2}\right)$$

with principal value given by

$$\begin{aligned} \exp\left(-\left(\frac{\pi}{4}\right) + i\ln\sqrt{2}\right) &= e^{-\pi/4}e^{i\ln\sqrt{2}} \\ &= e^{-\pi/4}\cos(\ln\sqrt{2}) + ie^{-\pi/4}\sin(\ln\sqrt{2}) \end{aligned}$$

B6 (i) This is a standard definition from the course: if $\gamma(t)$, $a \le t \le b$, is a parametrisation of γ then

$$\int_\gamma f = \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

(ii) The Fundamental Theorem of Contour Integration is Theorem 4.3.3 in the notes. It says that:

> Suppose $f : D \to \mathbb{C}$ is continuous and suppose that f has an anti-derivative F on D (so that $F'(z) = f(z)$ for all $z \in D$). Let γ be a contour from z_0 to z_1 in D. Then $\int_\gamma f = F(z_1) - F(z_0)$.

Some of you wrote down the wrong theorem (the Residue Theorem, or algebraic properties of the contour integral such as $\int_{-\gamma f} = -\int_\gamma f$) as the statement of the FToCI; this isn't what the question is asking you.

Very few of you gave the proof of this. The trick is to introduce a new function $W(t) = F(\gamma(t))$ and note that $W'(t) = f(\gamma(t))\gamma'(t)$ by the chain rule. One can then split $W(t)$ into its real and imaginary parts and use the Fundamental Theorem of Calculus to calculate $\int_\gamma f$.

(iii) The point the question is asking for is that: if γ is a closed loop (so that it starts and ends at the same point, say z_0) and if f has an antiderivative F then, by the FToCI, $\int_\gamma f = F(z_0) - F(z_0) = 0$.

To calculate the integral, the first thing to do is to write down a parametrisation of C. One parametrisation that works is $C(t) =$

$1+2e^{it}$, $0 \le t \le 2\pi$. Hence we can calculate

$$\begin{aligned}
\int_C g &= \int_0^{2\pi} g(C(t))C'(t)\,dt \\
&= \int_0^{2\pi} \frac{1+2e^{it}}{2e^{it}} \times 2ie^{it}\,dt \\
&= \int_0^{2\pi} i(1+2e^{it})\,dt \\
&= \int_0^{2\pi} i\,dt + \int_0^{2\pi} e^{it}\,dt \\
&= it + \frac{2}{i}e^{it}\Big|_0^{2\pi} \\
&= 2\pi i \ne 0.
\end{aligned}$$

Common mistakes here were the following: (i) writing $C(t) = 2e^{it}$ (this has the wrong centre), (ii) writing $g(z) = 1/(z-1)$ not $z/(z-1)$, (iii) not evaluating the integral correctly between the limits. For the latter point, note that when $t = 0$, we have that $e^{it} = 1$ (not 0). (Also note that $e^{2\pi i} = 1$ so one can simplify $e^{2\pi i} - e^0 = 0$.)

By the FToCI, if g had an anti-derivative then, as C is a closed loop, $\int_C g$ would be 0. Hence g does not have an anti-derivative on any domain that contains C.

B7 (i) f has a singularity at z_0 if f is not differentiable at z_0. (Many of you said that the definition was that f is not defined at z_0; I repeatedly said in the lectures that, although this is (for us, in this course) how singularities arise, this is *not* the definition.)

The point z_0 is an isolated singularity if there exists $r > 0$ such that f is differentaible on $0 < |z - z_0| < r$ (aside: this means that there are no other singularities 'near' z_0).

Suppose f has an isolated singularity at z_0. Then, by Laurent's Theorem, we can expand f as a Laurent series on the annulus $0 < |z - z_0| < r$:

$$\sum_{n=1}^{\infty} b_n(z-z_0)^{-n} + \sum_{n=0}^{\infty} a_n(z-z_0)^n.$$

The first part (involving the b_ns) is the principal part of the Laurent series. If the principal part contains no terms then z_0 is a removable singularity. If the principal part contains infinitely many terms then z_0 is an isolated essential singularity. If the principal part contains finitely many terms (and the most negative power that occurs is m) then z_0 is a pole of order m.

If z_0 is a pole of order m then $\mathrm{Res}(f, z_0) = b_1$, the coefficient of $(z - z_0)^{-1}$. Several of you wrote down a formula (either Lemma 7.4.1 (which only works for simple poles) or Lemma 7.4.2) for the residue; note that this is not the definition!

(ii) We know that

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots + z^n + \cdots$$

for $|z| < 1$ (think geometric progression) and that we can differentiate a power series term-by-term inside the disc of convergence. Hence, differentiating the above, we have

$$\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + 4z^3 + \cdots + nz^{n-1} + \cdots$$

for $|z| < 1$.

(iii) Using (ii), for $0 < |z| < 1$, we have

$$\begin{aligned} \frac{1}{z(1-z)^2} &= \frac{1}{z}\left(1 + 2z + 3z^2 + \cdots\right) \\ &= \frac{1}{z} + 2 + 3z + \cdots \end{aligned}$$

so that g has a simple pole with residue 1 at $z = 0$.
Let $w = z - 1$ so that $z = 1 + w$. Then (summing a geometric progression with common ratio $-w$)

$$\begin{aligned} g(z) &= \frac{1}{(1+w)w^2} = \frac{1}{w^2(1-(-w))} \\ &= \frac{1}{w^2}\left(1 - w + w^2 - w^3 + \cdots\right) \\ &= \frac{1}{(z-1)^2} - \frac{1}{(z-1)} + 1 - (z-1) + \cdots. \end{aligned}$$

Hence g has a pole of order 2 with residue -1 at $z = 1$.

(iv) This is similar to Exercise 6.7; very few of you attempted this. By the Estimation Lemma, we have that for any $r < R$

$$\begin{aligned} |b_n| &\leq \frac{1}{2\pi} \times \sup_{z \in C_r} |f(z)| \times |z|^{n-1} \times 2\pi r \\ &\leq \frac{M}{r} \times r^{n-1} \times r = Mr^{n-1}. \end{aligned}$$

If $n \geq 2$ then this converges to 0 as $r \to 0$. Hence $b_n = 0$ if $n \geq 2$. Hence f has Laurent series

$$\frac{b_1}{z - z_0} + a_0 + \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

and so has either a simple pole at z_0 (if $b_1 \neq 0$) or a removable singularity at z_0 (if $b_1 = 0$).

The two possibilities occur. Take $f(z) = 1/z$ and $f(z) = 1$. Then in both cases $|f(z)| \leq M/|z|$ for all $z \neq 0$.

B8 (i) Let $f(z) = 1/(3z^2 + 10iz - 3)$. Then f has singularities where the denominator vanishes. Note that $3z^2 + 10iz - 3 = (z+3i)(3z+i)$ (you can either work this out by inspection, or use the quadratic formula, or use the fact that the question tells you where the poles are and work backwards). Hence f has simple poles at $z = -3i, -i/3$.

Hence, using the formula for the residue that is given in the question,

$$\begin{aligned}
\mathrm{Res}(f, -i/3) &= \lim_{z \to -i/3} (z + i/3) \times \frac{1}{(z+3i)(3z+i)} \\
&= \lim_{z \to -i/3} \frac{1}{3} \times (3z+i) \times \frac{1}{(z+3i)(3z+i)} \\
&= \lim_{z \to -i/3} \frac{1}{3} \times \frac{1}{3z+i} \\
&= \frac{1}{3} \times \frac{1}{\left(\frac{-i}{3} + 3i\right)} = \frac{1}{8i}.
\end{aligned}$$

(Note that it is very easy to lose a factor of 3 in this.)

Similarly,

$$\begin{aligned}
\mathrm{Res}(f, -3i) &= \lim_{z \to -3i} (z+3i) \times \frac{1}{(z+3i)(3z+i)} \\
&= \frac{-1}{8i}.
\end{aligned}$$

(ii) Cauchy's Residue Theorem is Theorem 7.3.1 in the notes. Note that the Residue Theorem works functions with poles of any order, not just simple poles.

(iii) Let $z = e^{it}$. Then $dz = ie^{it}\,dt = iz\,dt$. As t varies from 0 to 2π, z goes around C_1 once anticlockwise. Also remember that

$$\sin t = \frac{e^{it} - e^{-it}}{2i} = \frac{z - z^{-1}}{2i}$$

(remember that it's a $2i$ not a 2 in the denominator).

Hence

$$\int_0^{2\pi} \frac{1}{5 + 3\sin t}\,dt = \int_{C_1} \frac{1}{5 + 3\left(\frac{z - z^{-1}}{2i}\right)} \frac{dz}{iz}$$

$$\begin{aligned} &= 2\int_{C_1} \frac{1}{z(10i + 3z - 3z^{-1})}\,dz \\ &= 2\int_{C_1} f(z)\,dz \end{aligned}$$

(be careful with is and minus signs here).
The only pole of f that lies inside C_1 is the pole at $-i/3$. By the Residue Theorem we have that

$$\int_{C_1} f(z)\,dz = 2\pi i\,\mathrm{Res}(f, -i/3) = 2\pi i \times 1/8i = \pi/4.$$

Hence

$$\int_0^{2\pi} \frac{1}{5 + 3\sin t}\,dt = 2 \times \frac{\pi}{4} = \frac{\pi}{2}.$$

If we try to use the same method to evaluate $\int_0^{2\pi} 1/(3+3\sin t)\,dt$ then we see that

$$\int_0^{2\pi} \frac{1}{3 + 3\sin t}\,dt = \frac{2}{3}\int_{C_1} \frac{dz}{z^2 + 2iz - 1} = \frac{2}{3}\int_{C_1} \frac{dz}{(z+i)^2}.$$

This integrand has a singularity at $-i$, which is on C_1. The Residue Theorem doesn't apply when there are singularities on the contour, and so we cannot use this method to calculate $\int_0^{2\pi} 1/(3+3\sin t)\,dt$. (As an aside, one can prove that $\int_0^{2\pi} 1/(3 + 3\sin t)\,dt$ is infinite.)

Charles Walkden
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