Feedback on MATH20101 Real and Complex Analysis (Complex Analysis part)

Overall comments. I was very pleased by how well many of you did on the complex analysis part of the exam for MATH20101. If you have any comments on the course (particularly how I do the tutorial classes and use Kahoot) then please let me know.

B5

(i) These are standard definitions from the course: $\partial g/\partial x(x_0, y_0)$ exists if

$$\lim_{h\to 0} \frac{g(x_0+h, y_0) - g(x_0, y_0)}{h}$$

exists. f is differentiable at $z_0 \in D$ if $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists. f is holomorphic on D is f is differentiable at $z_0 \in D$ for all $z_0 \in D$.

(ii) This is part of the proof of the Cauchy-Riemann theorem. The Cauchy Riemann theorem involves calculating $f'(z_0)$ in two ways, first by setting $z = (x_0 + h) + iy_0$ and then letting $h \to 0$ and second by setting $z = x_0 + i(y_0 + k)$ and letting $k \to 0$; the question is asking you to the second case. See the proof of Theorem 2.5.1 in the notes and the calculation in equation (2.5.3) in particular (and note how careful you need to be with minus signs and the fact that $1/i = -i$).

(iii) The Cauchy-Riemann Theorem says that **if** f is differentiable at $z_0 = x + o + iy_0$ **then** then partial derivatives of u, v exist at (x_0, y_0) and the Cauchy-Riemann equations hold.

The partial converse (and what the question is asking you to write down) is that **if** the partial derivatives of u, v exist at (x_0, y_0), and the Cauchy-Riemann equations hold, **and** the partial derivatives are continuous at (x_0, y_0) **then** f is differentiable at z_0.

(iv) This is similar to Exercise 2.10. Write $f(x + iy) = u(x, y) + iv(x, y)$. Suppose $v(x, y) = c$, a constant. Then $\partial v/\partial x = 0$. As f is differentiable at $x + iy$, the Cauchy-Riemann equations hold. Hence $\partial u/\partial y = 0$. Hence $u(x, y) = \alpha(x)$, for some function $\alpha(x)$ depending only on x.

Similarly, $\partial v/\partial y = 0$. Hence $\partial u/\partial y = 0$ so that $u(x, y) = \beta(y)$, for some function $\beta(y)$ depending only on y.

Hence $u(x, y) = \alpha(x) = \beta(y)$ and the only possibility is that $u(x, y)$ is equal to a constant. Hence f is equal to a constant.

(v) Let $f(x + iy) = (xy^2)^{1/3}$ so that $u(x, y) = (xy^2)^{1/3}$ and $v(x, y) = 0$. To calculate the partial derivatives at $(0, 0)$ we need to do it from first principles (any attempt to do it via the algebraic rules of differentiation is doomed to failure). Note that

$$\frac{\partial u}{\partial x}(0,0) = \lim_{h\to 0}\frac{u(h,0)-u(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$$

and similarly $\partial u/\partial y(0,0) = 0$. As $v(x,y) = 0$, it's clear that $\partial v/\partial x(0,0) = \partial v/\partial y(0,0) = 0$. Hence the Cauchy-Riemann equations hold at $(0,0)$.

However, f is not differentiable at 0. Let h be real and consider $z = h + ih$. Then $z \to 0$ as $h \to 0$ and

$$\lim_{h\to 0}\frac{f(h+ih)-f(0)}{h+ih-0} = \lim_{h\to 0}\frac{(hh^2)^{1/3}-0}{h+ih} = \lim_{h\to 0}\frac{h}{h+ih} = \frac{1}{1+i}.$$

However if we let $z = h$ and note that $z \to 0$ as $h \to 0$ then

$$\lim_{h\to 0}\frac{f(h)-f(0)}{h-0} = 0.$$

If f were differentiable at 0 then these two limits would be the same; as they are not, then f is not differentiable at 0.

This does not contradict the Cauchy-Riemann Theorem as the partial derivatives are continuous at the origin.

B6 (i) There are a few ways of doing the algebra here (and it really is just algebraic manipulations). For example

$$\begin{aligned}
\sin z &= \sin(x+iy) \\
&= \frac{1}{2i}\left(e^{i(x+iy)} - e^{-i(x+iy)}\right) \\
&= \frac{1}{2i}\left(e^{-y}e^{ix} - e^{y}e^{-x}\right) \\
&= \frac{1}{2i}\left(e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)\right) \\
&= \frac{1}{2}\left((e^{y} - e^{-y})\sin x\right) + \frac{1}{2i}\left((-e^{y} + e^{-y})\cos x\right) \\
&= \sin x\cosh y + i\cos x\sinh y
\end{aligned}$$

(note that you need to be very careful with minus signs here).

(ii) By (i), $\sin z = 0$ if and only if

$$\begin{aligned}
\sin x\cosh y = 0 \qquad (1) \\
\cos x\sinh y = 0 \qquad (2).
\end{aligned}$$

As $\cosh y > 0$ for all $y \in \mathbb{R}$, (1) implies that $\sin x = 0$. We know that the real zeros of sin occur at the integers, hence $x = k\pi$ for some $k \in \mathbb{Z}$. From (2), $\cos k\pi \sinh y = (-1)^k \sinh y = 0$. Hence $\sinh y = 0$. The only real zero of sinh is at 0, hence $y = 0$.

(iii) Write $z = x + iy$ and let p be a period for sin. Put $z = 0$ in $\sin(z+p) = \sin z$ to see that $\sin p = \sin 0 = 0$. Hence, by (ii), $\sin p = k\pi$ for some $k \in \mathbb{Z}$. Now

$$\begin{aligned} \sin(z+n\pi) &= \sin(z+(n-1)\pi+\pi) \\ &= \sin(z+(n-1)\pi)\cos\pi + \cos(z+(n-1)\pi)\sin\pi \\ &= -\sin(z+(n-1)\pi). \end{aligned}$$

Inductively, we have $\sin(z+n\pi) = (-1)^n \sin z$. Hence $\sin(z + n\pi) = \sin z$ if and only if n is even. Hence $p = 2k\pi$, $k \in \mathbb{Z}$. Suppose $e^{z+p} = e^z$ for all $z \in \mathbb{C}$. Again, putting $z = 0$ gives $e^p = e^0 = 1$. Let $p = x + iy$. Then

$$e^p = e^x e^{iy} = e^x(\cos y + i \sin y) = 1.$$

Hence

$$\begin{aligned} e^x \cos y = 1 \qquad (3) \\ e^x \sin y = 0 \qquad (4) \end{aligned}$$

As $e^x > 0$ for all $x \in \mathbb{R}$ we have $\sin y = 0$, i.e. $y = n\pi$, $n \in \mathbb{Z}$. Hence $e^x \cos n\pi = e^x(-1)^n = 1$. This has no real solutions when n is odd. When n is even, we have $e^x = 1$, hence $x = 0$. Hence the periods of exp are $2k\pi i$, $k \in \mathbb{Z}$.

(iv) From the calculations in (iii) we have that $e^w = 1$ if and only if $w = 2k\pi i$, $k \in \mathbb{Z}$. Now $e^z = e^{iz}$ iff $e^{z-iz} = 1$ iff $z - iz = 2k\pi i$ ($k \in \mathbb{Z}$) iff $z = 2k\pi i/(1-i)$ ($k \in \mathbb{Z}$).

B7 (i) If $z \in A, B, C, D$ then $w(\gamma, z) = 1, 2, 3, 0$, respectively.

(ii) This is a standard definition from the course: $\int_\gamma f = \int_a^b f(\gamma(t))\gamma'(t)\,dt$.

(iii) A parametrisation of γ_1 is given by

$$\gamma_1(t) = 2 + 3i + 2e^{it},\ 0 \le t \le 2\pi.$$

(There are other possible answers, for example $2 + 3i + 2e^{2\pi i t}$, $0 \le t \le 1$. Many of you forgot that the circle has radius 2 and wrote something like $2+3i+e^{it}$.) To calculate the winding number explicitly, just put this parametrisation into the definition in (ii):

$$\begin{aligned} w(\gamma_1, 2+3i) &= \frac{1}{2\pi i}\int_{\gamma_1} \frac{dz}{z-(2+3i)} \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{1}{(2+3i)+2e^{it}-(2+3i)} \times 2ie^{it}\,dt \\ &= \frac{1}{2\pi i}\int_0^{2\pi} i\,dt = 1 \end{aligned}$$

(as $\gamma_1'(t) = 2ie^{it}$).

(iv) The first part is bookwork: stating the Generalised Cauchy Theorem. (Be careful about the fact that the hypotheses involve points *not* in D.) For the second part, note that the only point not in D is $2+3i$. Clearly $w(-\gamma_2, 2+3i) = -2$. Hence

$$w(\gamma_1, z) + w(\gamma_1, z) + w(-\gamma_2, z) = 1 + 1 - 2 = 0$$

for all points $z \notin D$. So by the GCT

$$\int_{\gamma_1} f + \int_{\gamma_1} f + \int_{-\gamma_2} f = 0.$$

Rearranging this gives that

$$\int_{\gamma_2} f = 2 \int_{\gamma_1} f = 2 + 2i.$$

B8 (i) We discussed an example of this in the course. Take $f(x) = x$. Then

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} x\,dx = \lim_{R\to\infty} \frac{1}{2}(R^2 - R^2) = 0$$

but

$$\lim_{A,B\to\infty} \int_{-A}^{B} f(x)\,dx = \lim_{A,B\to\infty} \int_{-A}^{B} \frac{1}{2}(B^2 - A^2)$$

doesn't exist. The criterion for both limits to exist and to have the same value is: there exists $K > 0$, $C > 0$, $r > 1$ such that $|f(x)| < C/|x|^r$ for $|x| > K$. (As an aside: the criterion doesn't work if one only assumes $r \geq 1$ but we didn't discuss this in the lectures.)

(ii) As the function isn't defined, and so isn't differentiable, when the denominator vanishes (but is differentiable everywhere else), we see that f has poles at $\pm 2i, \pm 3i$. As these are simple zeros of the denominator, these are simple poles of f.It's straightforward from the formula $\text{Res}(f, z_0) = \lim_{z\to z_0}(z - z_0)f(z)$ (when z_0 is a simple pole) to see that $\text{Res}(f, 2i) = -1/5i$ and $\text{Res}(f, 3i) = 3/10i$ (be careful with minus signs!). If $R > 3$ then it follows from the Residue Theorem that $\int_{\Gamma_R} f = 2\pi i(\text{Res}(f, 2i) + \text{Res}(f, 3i)) = \pi/5$. The argument then follows that which we saw in the lectures/support classes. First there's an Estimation Lemma argument. Let $z \in S_R$. Then $|z| = R$. By the reverse triangle ineq we have

$$|z^2 + 4| = |z^2 - (-4)| \geq |z|^2 - 4 = R^2 - 4, \text{ if } R > 2,$$

so that

$$\frac{1}{|z^2+4|} \leq \frac{1}{R^2-4}.$$

(Many of you tried to argue that $|z^2+4| \leq |z^2|+4 \leq R^2+4$ so that $\frac{1}{|z^2+4|} \leq \frac{1}{R^2+4}$, which doesn't work). Similarly,

$$\frac{1}{|z^2+4|} \leq \frac{1}{R^2-9}$$

if $R > 3$. Hence if $R > 3$ then

$$|f(z)| \leq \frac{R^2}{(R^2-4)(R^2-9)}.$$

Note that the length of S_R is πR (not $2\pi R$). By the Estimation lemma,

$$\left| \int_{S_R} f \right| \leq \frac{R^2}{(R^2-4)(R^2-9)} \times \pi R \to 0$$

as $R \to \infty$. Also note that $|f(x)| \leq x^2/(x^2-4)(x^2-9) \leq 1/x^2$ so the technical hypothesis holds and the infinite integral exists and equals the principle value of the integral. Hence

$$\frac{\pi}{5} = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx + \lim_{R\to\infty} \int_{S_R} f = \int_{-\infty}^{\infty} f(x)\,dx.$$

(iii) There were some very creative and ingeneous attempts to work out this integral, some of which worked. The observation I intended you to make was that $f(x) = x/(x^2+4)(x^2+9)$ is an odd function, so $f(-x) = -f(x)$. Hence

$$\int_{-R}^{R} f(x)\,dx = 0$$

and letting $R \to \infty$ shows that $\int_{-\infty}^{\infty} f(x)\,dx = 0$.