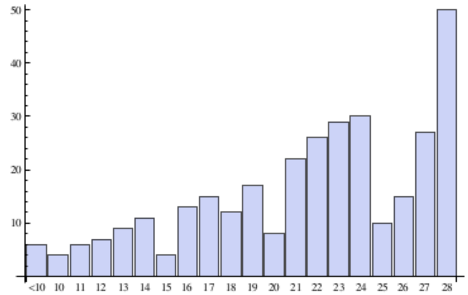
Solutions for MATH20101 Real & Complex Analysis, Complex Analysis coursework test, October 2018

The total number of marks available was 28. The average mark was 22. The distribution of marks is shown opposite.



Question Q1(i)

Learning Outcome

Use power series to define a holomorphic function and calculate its radius of convergence.

Solution

(i) We use the fact that if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ exists and is equal to $1/R$ then R is the radius of convergence (convention: $1/0=\infty$, $\infty = 1/0$).

(a) Let $a_n = n^3$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^3}{n^3} = \left(1+\frac{1}{n}\right)^3 \to 1 = \frac{1}{R}$$

as $n\to\infty$. Hence $R=1$.

[2 marks]

Feedback. Most people answered this correctly. Some people tried writing

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^3 z^{n+1}}{n^3 z^n}\right|$$

and got a z in the formula for R. The radius of convergence is a number; it cannot depend on z. Another common mistake was to write $a_n = n^3$ so $a_{n+1} = n^4$.

(b) Let $a_n = i^n/3^n$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{i^{n+1}3^n}{i^n 3^{n+1}}\right| = \frac{1}{3} \to \frac{1}{3} = \frac{1}{R}$$

as $n\to\infty$. Hence $R=3$.

[2 marks]

Feedback. Using the 'root test' formula for R also works. A very significant number of people wrote $R = 3/i$; this happens if you forget the modulus signs above. The radius of convergence must be a (non-negative) real number.

Question Q1(ii)

Learning Outcome

Define and perform computations with elementary holomorphic functions such as sin, cos, sinh, cosh, exp, log, and functions defined by power series.

Solution

Let $w = x+iy$ and suppose $e^w = z$. Then $z = e^{x+iy} = e^x e^{iy}$. Taking the modulus gives $|z| = e^x$, hence $x = \ln|z|$. Taking the argument gives $y = \arg(z)$.

[4 marks]

Feedback. This is bookwork (see §3.4.5 in the online notes). See also my email regarding the typographical mistake on the paper.

Question Q1(iii)

Learning Outcome

Define and perform computations with elementary holomorphic functions such as sin, cos, sinh, cosh, exp, log, and functions defined by power series.

Solution

We have

$$|-1+i| = \sqrt{2},\ \mathrm{Arg}(-1+i) = 3\pi/4$$
$$|i| = 1,\ \mathrm{Arg}\, i = \pi/2.$$

Hence

$$\mathrm{Log}(-1+i) = \ln\sqrt{2} + \frac{3\pi i}{4},\ \mathrm{Log} i = \ln 1 + \frac{\pi i}{2} = \frac{\pi}{2}.$$

Note that

$$\begin{aligned}\mathrm{Log}(-1+i)i &= \mathrm{Log} -1 - i = \ln\sqrt{2} - \frac{3\pi i}{4} \\ &\neq \mathrm{Log}(-1+i) + \mathrm{Log} i.\end{aligned}$$

[6 marks]

Feedback. Almost everybody correctly determined $\mathrm{Log}(-1+i)$ and $\mathrm{Log} i$. Several of you were determining the argument by calculating $\tan^{-1} y/x$. It's much easier (and safer, given that tan is π-periodic, not 2π-periodic) in these cases where the argument is at 45 degrees to one of the axes to sketch a picture and from that write down immediately what the argument is.

For the second part, many people picked different examples of z_1, z_2. In some cases, these worked; in others, they did not. The phenomena you want to capture is that, in general, $\mathrm{Arg}\, z_1 z_2 \neq \mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2$. This happens when $\mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2 \notin (-\pi, \pi]$. Taking $z_1 = -1+i, z_2 = i$ works (see above); obviously if you picked another example that worked then you got full credit. Other examples, such as $z_1 = z_2 = i$ do not (here $\mathrm{Arg}\, z_1 = \pi/2$ so $\mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2 = \pi = \mathrm{Arg}\, z_1 z_2$. Note that $\mathrm{Log} -1 = \ln|-1| + i\,\mathrm{Arg}(-1) = 0 + i\pi$).

Question Q2(i)

Learning Outcome

define the complex integral and use a variety of methods (the Fundamental Theorem of Contour Integration, Cauchys Theorem, the Generalised Cauchy Theorem and the Cauchy Residue Theorem) to calculate the complex integral of a given function.

Solution

γ has $\gamma(t) = 3 + i + 2e^{it}$, $0 \le t \le 2\pi$, as a parametrisation.

Hence

$$\begin{aligned}\int_\gamma \frac{z}{z-3+i}\, dz &= \int_0^{2\pi} \frac{\gamma(t)}{\gamma(t)-(3+i)}\gamma'(t)\, dt \\ &= \int_0^{2\pi} \frac{3+i+2e^{it}}{2e^{it}} \times 2ie^{it}\, dt \\ &= i\int_0^{2\pi} 3+i+2e^{it}\, dt \\ &= 2\pi i(3+i)\end{aligned}$$

(as $\int_0^{2\pi} e^{it}\, dt = 0$).

[8 marks]

Feedback. In general, most people answered this very well.

Quite a few people wrote $\gamma(t) = 3 + i + 2e^{it}$ for the parametrisation without giving the values of t. You do need to do this. Different values of t give different paths. For example, $\gamma(t) = 3 + i + 2e^{it}$, $0 \le t \le \pi$, determines the top-half of the semi-circle with centre $3+i$ radius 2.

To calculate the integral, you need (for the moment, until we do Cauchy's Residue Theorem at the end of the course) to do this from the definition, as above. Note that you can't write

$$\int_\gamma \frac{z}{z-(3+i)}\, dz = \int_\gamma \frac{1}{z-(3+i)}\, dz,$$

because this isn't true (if you change the integrand then you change the integral!).

Some of you tried to do this using coss and sins instead by writing $\gamma(t) = 3 + 2\cos t + i(1 + 2\sin t)$. This will work out, but is notationally more complicated (as it's harder to see that you can cancel a $\cos t + i\sin t$ in the numerator and denominator).

Question Q2(ii)

Learning Outcome

define the complex integral and use a variety of methods (the Fundamental Theorem of Contour Integration, Cauchys Theorem, the Generalised Cauchy Theorem and the Cauchy Residue Theorem) to calculate the complex integral of a given function.

Solution

The Fundamental Theorem of Contour Integration (FToCI): Suppose that $f : D \to \mathbb{C}$ is continuous and has an antiderivative F on D. Let γ be a smooth path in D which starts at z_0 and ends at z_1. Then

$$\int_\gamma f = F(z_1) - F(z_0).$$

[2 marks]

Feedback. A very common error here is to think that the FToCI says:

$$\int_{\gamma_1+\cdots+\gamma_n} f = \int_{\gamma_1} f + \cdots + \int_{\gamma_n} f.$$

(This has happened before when 'State the FToCI' has appeared as part of an exam question; I'm genuinely puzzled as to why and if anyone can shed any light on this then I'd be interested to hear.)

Another common error is to write

$$\int_\gamma f = F(b) - F(a).$$

This isn't right. If you want to write the FToCI in terms of a parametrisation of γ then you would need to write: Suppose that $f : D \to \mathbb{C}$ is continuous and has an antiderivative F on D. Let $\gamma; [a,b] \to D$ be a contour in D. Then

$$\int_\gamma f = F(\gamma(b)) - F(\gamma(a)).$$

(This is because $\gamma(a), \gamma(b) \in D$ are the end points of D. If you did write something like: $\int_\gamma f = F(b) - F(a)$ where $a, b \in D$ are the end points of γ then I gave you the benefit of the doubt.)

As a general comment, many of you just wrote $\int f = F(z_1) - F(z_0$ as the statement of the FToCI. To state a theorem, you need to make clear what the hypotheses are (here, that (i) $f : D \to \mathbb{C}$ is continuous and has an antiderivative F on D, (ii) γ is a contour in D with end points at z_0, z_1) and then make clear what the conclusions are (here, that $\int_\gamma f = F(z_1) - F(z_0)$). Stating just the conclusions without the hypotheses will lose you marks in general.

If you said 'smooth path' rather than 'contour' then I marked that as correct (as contours are comprised of finitely many smooth paths joined together at their endpoints, so the statement for contours follows immediately from the statement for smooth paths).

Question Q2(iii)

Learning Outcome

Define the complex integral and use a variety of methods (the Fundamental Theorem of Contour Integration, Cauchys Theorem, the Generalised Cauchy Theorem and the Cauchy Residue Theorem) to calculate the complex integral of a given function.

Solution

The function f as in (i) does not have an antiderivative on $\mathbb{C}\setminus\{3+i\}$. If it did, then $\int_\gamma f = 0$ for all closed loops γ by the FTCI. But in (i) we saw an example of a closed loop γ in D such that $\int_\gamma f \neq 0$.

[4 marks]

Feedback. Several of you wrote answers along the lines of 'f is holomorphic so f has an anti-derivative', or 'f is not continuous as it is not defined at $3 + i$ and so does not have an anti-derivative'. Neither of these are right. Firstly, most holomorphic functions do not have anti-derivatives. Secondly, we're looking at $f : \mathbb{C} \setminus \{3+i\} \to \mathbb{C}$, so the fact that f is not defined at $3+i$ is irrelevant.

Some of you tried to find (for example, using integration by parts) a possible anti-derivative in terms of a logarithm and then saying that Log is not differentiable. This can (with care on the domain you're working on) provide very strong circumstantial evidence that f does not have an anti-derivative, but it's not a rigorous proof (maybe there's a more complicated function, not involving Log that is an anti-derivative).