The University of Manchester

MATH20101

Real & Complex Analysis
Complex Analysis coursework test

In-class Examination
40 minutes

31st October 2018

10:00am – 10:40am

This test will count 10% towards your final mark.

Answer ALL of the questions in the answer books provided. A total of 28 marks are available.

No prepared notes of any kind or mathematical tables are to be brought into the examination room.

Electronic calculators may be used, provided that they cannot store text.

Q1 (i) Determine the radii of convergence of the following power series. You may use any standard results from the course provided that you state them clearly.

$$\text{(a)} \sum_{n=0}^{\infty} n^3 z^n, \qquad \text{(b)} \sum_{n=0}^{\infty} \frac{i^n}{3^n} z^n.$$

[4 marks]

(ii) Let $z \in \mathbb{C} \setminus \{0\}$. Suppose that $\exp w = z$. Let $w = x + iy$. Show that

$$x = \ln |z| \quad \text{and } y = \arg(z).$$

(You may *not* assume the definition of the complex logarithm.) [4 marks]

(iii) Recall that the principal value of the logarithm is defined to be

$$\operatorname{Log} z = \ln |z| + i \operatorname{Arg}(z)$$

where $\operatorname{Arg}(z) \in (-\pi, \pi]$ is the principal value of the argument of z.
Show that

$$\operatorname{Log}(-1 + i) = \ln(\sqrt{2}) + \frac{3\pi i}{4}, \qquad \text{and } \operatorname{Log} i = \frac{i\pi}{2}.$$

Hence give an example to show that, in general,

$$\operatorname{Log} z_1 + \operatorname{Log} z_2 \neq \operatorname{Log} z_1 z_2.$$

[6 marks]

Q2 Recall that if D is a domain, $\gamma : [a, b] \to D$ is a path in D and $f : D \to \mathbb{C}$ is continuous then the integral $\int_\gamma f$ is defined to be

$$\int_a^b f(\gamma(t))\gamma'(t)\, dt.$$

(i) Let γ denote a circular path with centre $3 + i$ and radius 2, described once anticlockwise. Write down a parametrisation of γ.
Hence show that

$$\int_\gamma \frac{z}{z - (3 + i)}\, dz = 2\pi i(3 + i).$$

[8 marks]

(ii) State, without proof, the Fundamental Theorem of Contour Integration.

[2 marks]

(iii) Does the function

$$f(z) = \frac{z}{z - (3 + i)}$$

defined on the domain $D = \mathbb{C} \setminus \{3 + i\}$ have an anti-derivative? Give reasons for your answer. [4 marks]

END of exam paper