Introduction to self-similar growth-fragmentations

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Lecture notes available at my personal webpage: https://sites.google.com/site/qshimath
Overview

1. Background: fragmentation processes
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2. Construction of growth-fragmentation processes
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2. Construction of growth-fragmentation processes
3. Properties of self-similar growth-fragmentations
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1. Background: fragmentation processes
2. Construction of growth-fragmentation processes
3. Properties of self-similar growth-fragmentations
4. Martingales in self-similar growth-fragmentations
1. Background: fragmentation processes
Background

- **Fragmentation**: “the process or state of breaking or being broken into fragments”.

Fragmentation phenomena can be observed widely in nature: biology and population genetics, aerosols, droplets, mining industry, etc.

The first studies of fragmentation from a probabilistic point of view are due to Kolmogorov [1941] and Filippov [1961].

The general framework of the theory of stochastic fragmentation processes was built mainly by Bertoin [2001, 2002]. See Bertoin [2006] for a comprehensive monograph.

Fragmentations are relevant to other areas of probability theory, such as branching processes, coalescent processes, multiplicative cascades and random trees.
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- Fragmentations are relevant to other areas of probability theory, such as branching processes, coalescent processes, multiplicative cascades and random trees.
Model: Self-similar fragmentation processes [Bertoin 2002]

- **Index of self-similarity:** $\alpha \in \mathbb{R}$.
- **Dislocation measure:** $\nu$ sigma-finite measure on $[1/2, 1)$, such that
  \[
  \int_{[1/2, 1)} (1 - y) \nu(dy) < \infty.
  \]
- For every $y \in [1/2, 1)$, a fragment of size $x > 0$ splits into two fragments of size $xy$, $x(1 - y)$ at rate $x^\alpha \nu(dy)$. 

\[\begin{tikzpicture}
  \node[shape=circle,fill=blue!50] (A) at (0,0) {}; \node[shape=circle,fill=blue!50] (B) at (1,0) {}; \node[shape=circle,fill=blue!50] (C) at (2,0) {}; \node[shape=circle,fill=blue!50] (D) at (3,0) {}; 
  \draw[dashed] (A) -- (B) -- (C) -- (D);
\end{tikzpicture}\]
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  \[ \int_{[1/2, 1)} (1 - y) \nu(dy) < \infty. \]
- For every $y \in [1/2, 1)$, a fragment of size $x > 0$ splits into two fragments of size $xy, x(1 - y)$ at rate $x^\alpha \nu(dy)$.
- Record the sizes of the fragments at time $t \geq 0$ by a measure on $\mathbb{R}_+$
  \[ X(t) = \sum_{i \geq 1} \delta X_i(t). \]

The process $X$ is a self-similar fragmentation (without erosion) with characteristics $(\alpha, \nu)$.  
Examples:

- **Splitting intervals** [BrennanDurrett1986]:
  - $U_i$: i.i.d. uniform random variables on $(0, 1)$, arrive one after another at rate 1.
  - At time $t$, $(0, 1)$ is separated into intervals $I_1(t), I_2(t), \ldots$
  - $F(t) := \sum_{i \geq 1} \delta_{I_i(t)}$ is a self-similar fragmentation with characteristics $(1, \nu)$, where $\nu(dx) = 2dx, \quad x \in \left[\frac{1}{2}, 1\right)$.
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- **Splitting intervals** [BrennanDurrett1986]:
  - \( U_i \): i.i.d. uniform random variables on \((0, 1)\), arrive one after another at rate 1.
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  - \( F(t) := \sum_{i \geq 1} \delta_{l_i(t)} \) is a self-similar fragmentation with characteristics \((1, \nu)\), where \( \nu(dx) = 2dx, \quad x \in [\frac{1}{2}, 1) \).

- **The Brownian fragmentation**
  - Normalized Brownian excursion: \( B : [0, 1] \to \mathbb{R}_+ \).
  - \( O(t) := \{ x \in (0, 1) : B(x) > t \} \).
  - \( F(t) := \sum_{I: \text{component of } O(t)} \delta_{|I|} \)
  - \( F \) is a self-similar fragmentation with characteristics \((-\frac{1}{2}, \nu_B)\), where \( \nu_B(dx) = \frac{2}{\sqrt{2\pi x^3(1-x)^3}}dx, \quad x \in [\frac{1}{2}, 1) \).
2. Construction of growth-fragmentation processes
Growth-fragmentation processes

- (Markovian) growth-fragmentation processes [Bertoin 2017] describe the evolution of the sizes of a family of particles, which can grow larger or smaller with time, and occasionally split in a conservative manner.
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- Applications of the model: Random planar maps [Bertoin&Curien&Kortchemski, 2015+; Bertoin&Budd&Curien&Kortchemski, 2016+]
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- Simulation by I. Kortchemski & N. Curien: https://www.normalesup.org/ kortchem/images/tribord.gif
Construction of growth-fragmentations [Bertoin 2017]

- Starfishes:

  - Growth: The size of the ancestor evolves according to a positive self-similar Markov process (pssMp) $X$, with only negative jumps.
  
  - Regeneration: At each jump time $t \geq 0$ with $-y := X(t) - X(t-)$ < 0, a daughter is born with initial size $y$. The size evolution of the daughter has the same distribution as $X$ (but started at $y$), and is independent of other daughters.

  - Granddaughters are born at the jumps of each daughter, and so on.
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Simulation by B. Dadoun
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Simulation by B. Dadoun
Record the sizes of all individuals at time $t \geq 0$ by a point measure on $\mathbb{R}_+$:

$$X(t) := \sum_{u \in \mathcal{U}} \delta \chi_u(t-b_u) 1\{b_u \leq t, \chi_u(t-b_u) > 0\}, \quad t \geq 0.$$
Self-similar growth-fragmentations

- The population is indexed by the Ulam-Harris tree: \( U = \bigcup_{i=0}^{\infty} \mathbb{N}^i \). By convention \( \mathbb{N}^0 = \{\emptyset\} \).
  An element \( u = (n_1, \ldots, n_i) \in \mathbb{N}^i \), then \( uk := (n_1, \ldots, n_i, k) \) for \( k \in \mathbb{N} \).
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- Construct a **cell system**, that is a family

$$\left( \mathcal{X}_u, b_u \right), \quad u \in \mathcal{U}.$$

$\mathcal{X}_u$: evolution of the size of $u$ as time grows.

$b_u$: birth time of $u$. 
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1. Initialization: $b_\emptyset := 0$, $\mathcal{X}_\emptyset \overset{d}{=} P_x$ (the law of the pssMp $X$ started from $x$).
2. Induction: enumerate the jump times of $\mathcal{X}_u$ by $(t_k, k \in \mathbb{N})$ and let $y_k := -\Delta \mathcal{X}_u(t_k) > 0$. Then

$$b_{u_k} = b_u + t_k, \quad \mathcal{X}_{u_k} \overset{d}{=} P_{y_k}, \quad k \in \mathbb{N}.$$

The daughters $(\mathcal{X}_{u_k}, k \in \mathbb{N})$ are independent.
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Construct a **cell system**, that is a family

\[ (\mathcal{X}_u, b_u), \quad u \in U. \]

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\[ b_{uk} = b_u + t_k, \quad \mathcal{X}_{uk} \overset{d}{=} P_{y_k}, \quad k \in \mathbb{N}. \]

The daughters \( (\mathcal{X}_{uk}, k \in \mathbb{N}) \) are independent.

\((\mathcal{X}_u, u \in U)\) is a **Crump-Mode-Jagers branching process** [Jagers, 1983].

The law of \((\mathcal{X}_u, u \in U)\) is determined by the pssMp \( X \).
Record the sizes of all individuals alive at time $t \geq 0$ by a point measure on $\mathbb{R}_+$:

$$X(t) := \sum_{u \in U} \delta_{X_u(t-b_u)}1\{b_u \leq t, X_u(t-b_u) > 0\}, \quad t \geq 0,$$

where $\delta$ denotes the Dirac measure. The process $X$ is called a **self-similar growth-fragmentation** associated with $X$. 

$X$ does not contain any information of the genealogy. The law of the pssMp $X$ determines the law of $X$. 

**Question**

Is $X(t)$ a Radon measure on $\mathbb{R}_+$?
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Growth-Fragmentations
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Positive self-similar Markov processes

Let $X$ be a positive self-similar Markov process (pssMp) with no positive jumps, and $P_x$ be the law of $X$ starting from $X(0) = x$.

- $\alpha \in \mathbb{R}$ is the index of self-similarity: if $X(0) = x$, then for every $r > 0$

  $$(rX(r^{\alpha}t), t \geq 0)$$

  have the law of $P_{rx}$ and $P_x$. 

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- For the homogeneous case $\alpha = 0$: there exists a Lévy process $\xi$ such that $X^{(0)}(t) = \exp(\xi_t)$, $t \geq 0$. 

Quan Shi
Growth-Fragmentations
CIMAT, 11-15 December, 2017 14 / 34
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- For the homogeneous case $\alpha = 0$: there exists a Lévy process $\xi$ such that $X^{(0)}(t) = \exp(\xi_t), \quad t \geq 0$.

- $\xi$ (without positive jumps, not killed) is characterized by its Laplace exponent $\Phi : [0, \infty) \rightarrow \mathbb{R}$,

$$E \left[ e^{q\xi(t)} \right] = e^{\Phi(q)t}, \quad \text{for all } t, q \geq 0.$$

- The function $\Phi$ is convex, given by the Lévy-Khintchine formula:

$$\Phi(q) = \frac{1}{2} \sigma^2 q^2 + cq + \int_{(-\infty,0)} (e^{qx} - 1 + q(1 - e^x)) \Lambda(dx), \quad q \geq 0,$$

where $\sigma^2 \geq 0$, $c \in \mathbb{R}$, and the Lévy measure $\Lambda$ is a measure on $(-\infty, 0)$ with

$$\int_{(-\infty,0)} (|x|^2 \wedge 1) \Lambda(dx) < \infty.$$
Positive self-similar Markov processes

- Lamperti’s transform [Lamperti 1972]: when $\alpha \neq 0$:

$$X_t^{(\alpha)} = \exp(\xi_{\tau^{(\alpha)}(t)}), \quad 0 \leq t < \zeta^{(\alpha)},$$

where $(\tau^{(\alpha)}(t), t \geq 0)$ is an explicit time-change that depends on $\alpha$:

$$\tau^{(\alpha)}(t) := \inf \left\{ s \geq 0 : \int_0^s \exp(-\alpha \xi_r) dr > t \right\}, \quad t \geq 0.$$
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- The law of the pssMp $X$ is characterized by $(\Phi, \alpha)$. 
Lamperti’s time-change

\[ X^{(0)}(t) = e^{\xi(t)} \]

[\xi: \text{Lévy process}]

Simulation by B. Dadoun
Lamperti’s time-change

\[
X^{(0)}(t) = e^{\xi(t)} \quad [\xi: \text{Lévy process}]
\]

\[
X^{(\alpha)}(t) = X^{(0)} \left( \int_0^t (X^{(\alpha)}(s))^\alpha \, ds \right)
\]

Simulation by B. Dadoun
A coupling construction

Repeat Lamperti’s time-change for each trajectory from the homogeneous case:

\[ \mathcal{X}_u^{(\alpha)}(t) := \mathcal{X}_u^{(0)}(\tau_u^{(\alpha)}(t)) \]

with \( \tau_u^{(\alpha)}(t) := \inf \left\{ r \geq 0 : \int_0^r \mathcal{X}_u^{(0)}(s)^{-\alpha} ds \geq t \right\} \)

\( \alpha = 0 \)

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\[ \alpha = 0.5 \]

Simulation by B. Dadoun
3. Properties of self-similar growth-fragmentations
Cumulant function for the homogeneous case

- X: a homogeneous growth-fragmentation associated with a pssMp X with characteristics \((\Phi, \alpha = 0)\).
Cumulant function for the homogeneous case

- $X$: a homogeneous growth-fragmentation associated with a pssMp $X$ with characteristics $(\Phi, \alpha = 0)$.
- Define the cumulant $\kappa: [0, \infty) \to (-\infty, \infty]$ by

$$\kappa(q) := \Phi(q) + \int_{(-\infty, 0)} (1 - e^x)^q \Lambda(dx), \quad q \geq 0.$$
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  \kappa(q) := \Phi(q) + \int_{(\infty,0)} (1 - e^x)^q \Lambda(dx), \quad q \geq 0.
  \]
- $\kappa$ is convex and possibly takes value at $\infty$. But $\kappa(q) < \infty$ for all $q \geq 2$. 

Theorem (Bertoin, 2016)

Suppose that $\kappa(q) < \infty$. If $\alpha = 0$, then for every $t \geq 0$,

\[
E[X(t)] = x q \exp(\kappa(q)t).
\]
Cumulant function for the homogeneous case

- \( X \): a homogeneous growth-fragmentation associated with a pssMp \( X \) with characteristics \((\Phi, \alpha = 0)\).
- Define the cumulant \( \kappa: [0, \infty) \rightarrow (-\infty, \infty] \) by

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\kappa(q) := \Phi(q) + \int_{(-\infty,0)} (1 - e^x)^q \Lambda(dx), \quad q \geq 0.
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**Theorem (Bertoin, 2016)**

Suppose that \( \kappa(q) < \infty \). If \( \alpha = 0 \), then for every \( t \geq 0 \),

\[
E_x \left[ \int_{\mathbb{R}_+} y^q X_t(dy) \right] = x^q \exp(\kappa(q)t).
\]
Non-explosion condition

- $X$: a self-similar growth-fragmentation associated with a pssMp $X$ with characteristics $(\Phi, \alpha)$.
Non-explosion condition

- \( X \): a self-similar growth-fragmentation associated with a pssMp \( X \) with characteristics \((\Phi, \alpha)\).
- \( X \) is called **non-explosive**, if for every \( t \geq 0 \) and \( a > 0 \), \( X(t) \) has **finite** mass on \([a, \infty)\).
Non-explosion condition

- **X**: a self-similar growth-fragmentation associated with a pssMp $X$ with characteristics $(\Phi, \alpha)$.

- **X** is called **non-explosive**, if for every $t \geq 0$ and $a > 0$, $X(t)$ has finite mass on $[a, \infty)$.

- **[Bertoin&Stephenson, 2016]**: if $\alpha \neq 0$ and $\kappa(q) > 0$ for all $q \geq 0$, then $X$ explodes in finite time.
Non-explosion condition

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- $X$ is called non-explosive, if for every $t \geq 0$ and $a > 0$, $X(t)$ has finite mass on $[a, \infty)$.
- [Bertoin&Stephenson, 2016]: if $\alpha \neq 0$ and $\kappa(q) > 0$ for all $q \geq 0$, then $X$ explodes in finite time.

**Theorem (Bertoin, 2017)**

Suppose that there exists $q > 0$ such that $\kappa(q) \leq 0$. Then for every $t \geq 0$, there is the inequality

$$E_x \left[ \int_{\mathbb{R}^+} y^q X_t(dy) \right] \leq x^q.$$
Proof of the theorem

Using Lamperti’s transform, we have

\[ E_x \left[ X(t)^q + \sum_{0 \leq s \leq t} |\Delta X(s)|^q \right] \leq x^q, \quad \text{for all } t \geq 0. \]
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Consequence: fix \( t \geq 0 \), we have a supermartingale:

\[ \Sigma_n := \sum_{|u| \leq n-1, b_u \leq t} \mathcal{X}_u(t - b_u)^q + \sum_{|v| = n, b_v \leq s \leq t} |\Delta \mathcal{X}_v(s - b_v)|^q, \quad n \geq 1. \]
Proof of the theorem

- Using Lamperti’s transform, we have

$$E_x\left[X(t)^q + \sum_{0 \leq s \leq t} |\Delta X(s)|^q\right] \leq x^q, \quad \text{for all } t \geq 0.$$  

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$$\mathcal{E}_x\left[\sum_{u \in \mathbb{U} : b_u \leq t} \mathcal{X}_u(t - b_u)^q\right] \leq \mathcal{E}_x[\Sigma_\infty]$$

$$\leq \mathcal{E}_x[\Sigma_0] = \mathcal{E}_x\left[\mathcal{X}_0(t)^q + \sum_{0 \leq s \leq t} |\Delta \mathcal{X}_0(s)|^q\right] \leq x^q.$$
Properties of self-similar growth-fragmentations

Let $X$ be a self-similar growth-fragmentation driven by $X$ started from $X(0) = \delta_x$. Suppose that the non-explosion condition holds.

- [The branching property]
  Write $X(s) = \sum_{i \geq 1} \delta_{X_i(s)}$. Conditionally on $\sigma(X(r), r \leq s)$, we have
  \[
  (X(t + s), t \geq 0) \overset{d}{=} \sum_{i \geq 1} X^{(i)}(t),
  \]
  where $(X^{(i)}, i \geq 1)$ are independent self-similar growth-fragmentations driven by $X$, with $X^{(i)}$ started at $X^{(i)}(0) = \delta_{X_i(s)}$.

- $X$ is self-similar with index $\alpha$: $X(0) = \delta_x$. For every $\theta > 0$,
  \[
  (\sum_{i \geq 1} \delta_{\theta X_i(\theta^\alpha t)}, t \geq 0) \overset{d}{=} X \text{ started from } \delta_{\theta x}.
  \]
Extinction

Theorem (Bertoin 2017)

Suppose that $\alpha < 0$ and that there exists $q > 0$ such that $\kappa(q) < 0$. Then the extinction time

$$\inf\{t \geq 0 : X^{(\alpha)}(t) = 0\}$$

is $P_x$-a.s. finite for every $x > 0$. 
Theorem (Bertoin 2017)

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Proof.

$$\inf\{t \geq 0 : X^{(\alpha)}(t) = 0\} = \sup \left\{ b_u^{(\alpha)} + \zeta_u^{(\alpha)} : u \in \mathbb{U} \right\}$$
Theorem (Bertoin 2017)

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Proof.

- $\inf\{t \geq 0 : X^{(\alpha)}(t) = 0\} = \sup \left\{ b_u^{(\alpha)} + \zeta_u^{(\alpha)} : u \in U \right\}$
- $\mathcal{Y}_u^{(\alpha)}$: the ancestral lineage of $u$.
- Identity: $b_u^{(\alpha)} + \zeta_u^{(\alpha)} = \int_0^\infty \mathcal{Y}_u^{(0)}(s)^{-\alpha} ds$
Extinction

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Suppose that $\alpha < 0$ and that there exists $q > 0$ such that $\kappa(q) < 0$. Then the extinction time

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Proof.

- $\inf \{ t \geq 0 : X^{(\alpha)}(t) = 0 \} = \sup \left\{ b_u^{(\alpha)} + \zeta_u^{(\alpha)} : u \in U \right\}$
- $\mathcal{Y}_u^{(\alpha)}$: the ancestral lineage of $u$.
- Identity: $b_u^{(\alpha)} + \zeta_u^{(\alpha)} = \int_0^\infty \mathcal{Y}_u^{(0)}(s)^{-\alpha} \, ds$
- $\mathcal{Y}_u^{(0)}(t)^q \leq X_1^{(0)}(t)^q \leq e^{\kappa(q)t} \left( e^{-\kappa(q)t} \sum_{i \geq 1} X_i^{(0)}(t)^q \right) \leq Ce^{\kappa(q)t}$
4. Martingales in self-similar growth-fragmentations
Recall that $\kappa$ is convex, so it has at most two roots.
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Suppose that the Cramér’s hypothesis holds:

$$\omega_+ > \omega_0 > 0, \text{ s.t. } \kappa(\omega_-) = \kappa(\omega_+) = 0 \text{ and } \kappa'(\omega_-) > -\infty.$$  \[\text{[H]}\]
Recall that $\kappa$ is convex, so it has at most two roots.

Suppose that the Cramér’s hypothesis holds:

$$\omega_+ > \omega_- > 0, \text{ s.t. } \kappa(\omega_-) = \kappa(\omega_+) = 0 \text{ and } \kappa'(\omega_-) > -\infty.$$  \[H\]

[H] implies the non-explosion condition (there exists $q > 0$ with $\kappa(q) < 0$).
The genealogical martingales

- \((- \log X_u(0), u \in \mathbb{U})\) is a branching random walk.
- If \(\Phi(q) < 0\) and \(\kappa(q) < \infty\), then

\[
m(q) := E_1 \left[ \sum_{|u|=1} e^{-q(- \log X_u(0))} \right] = E_1 \left[ \sum_{0<s<\zeta} |\Delta X(s)|^q \right] = 1 - \frac{\kappa(q)}{\Phi(q)}.
\]

Lemma

Suppose that [H] holds.

1. \(M^+(n) := x^{-\omega} \sum_{|u|=n} X_u(0)^{\omega+}\)

is a martingale that converges \(\mathcal{P}_x\)-a.s. to 0.

2. For any \(p \in [1, \frac{\omega_+}{\omega_-})\), the martingale

\[
M^-(n) := x^{-\omega} \sum_{|u|=n} X_u(0)^{\omega-}
\]

converges \(\mathcal{P}_x\)-a.s. and in \(L^p(\mathcal{P}_x)\). Its terminal value \(M^-(\infty) > 0\).
Mellin transform of the potential measure

**Fact:** For a pssMp $X^{(\alpha)}$ with characteristics $(\Phi, \alpha)$:

$$E_x \left[ \int_0^\infty X^{(\alpha)}(t)^{q+\alpha} \, dt \right] = -\frac{1}{\Phi(q)} x^q. \quad \text{whenever } \Phi(q) < 0.$$
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**Proposition (BBCK, 2016+)**

For every $q$ with $\kappa(q) < 0$:

$$E_x \left[ \int_0^\infty \left( \sum_{i=1}^\infty X_i^{(\alpha)}(t)^{q+\alpha} \right) \, dt \right] = -\frac{1}{\kappa(q)} x^q.$$
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Remarks: Consider a growth-fragmentation $\tilde{X}$ associated with another pssMp $\tilde{X}$ with characteristics $(\tilde{\Phi}, \tilde{\alpha})$.

- $X \overset{d}{=} \tilde{X} \ (\Phi = \tilde{\Phi} \text{ and } \alpha = \tilde{\alpha}) \Rightarrow X \overset{d}{=} \tilde{X}$; but $X \overset{d}{=} \tilde{X} \not\Rightarrow X \overset{d}{=} \tilde{X}.$
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**Fact:** For a pssMp $X^{(\alpha)}$ with characteristics $(\Phi, \alpha)$:

$$E_X\left[ \int_0^{\infty} X^{(\alpha)}(t)^{q+\alpha} \, dt \right] = -\frac{1}{\Phi(q)} x^q. \quad \text{whenever } \Phi(q) < 0.$$

**Proposition (BBCK, 2016+)**

*For every $q$ with $\kappa(q) < 0$:

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**Remarks:** Consider a growth-fragmentation $\tilde{\mathbf{X}}$ associated with another pssMp $\tilde{\mathbf{X}}$ with characteristics $(\tilde{\Phi}, \tilde{\alpha})$.

- $X \overset{d}{=} \tilde{\mathbf{X}}$ $(\Phi = \tilde{\Phi}$ and $\alpha = \tilde{\alpha}) \Rightarrow X \overset{d}{=} \tilde{\mathbf{X}}$; but $X \overset{d}{=} \tilde{\mathbf{X}} \nRightarrow X \overset{d}{=} \tilde{\mathbf{X}}$.
- If $X \overset{d}{=} \tilde{\mathbf{X}}$, then $\kappa = \tilde{\kappa}$ and $\alpha = \tilde{\alpha}$.
Mellin transform of the potential measure

**Fact:** For a pssMp $X^{(\alpha)}$ with characteristics $(\Phi, \alpha)$:

$$E_x \left[ \int_0^\infty X^{(\alpha)}(t)^{q+\alpha} \, dt \right] = -\frac{1}{\Phi(q)} x^q, \quad \text{whenever } \Phi(q) < 0.$$ 

**Proposition (BBCK, 2016+)**

For every $q$ with $\kappa(q) < 0$:

$$E_x \left[ \int_0^\infty \left( \sum_{i=1}^\infty X_i^{(\alpha)}(t)^{q+\alpha} \right) \, dt \right] = -\frac{1}{\kappa(q)} x^q.$$ 

**Remarks:** Consider a growth-fragmentation $\tilde{X}$ associated with another pssMp $\tilde{X}$ with characteristics $(\tilde{\Phi}, \tilde{\alpha})$.

- $X \overset{d}{=} \tilde{X}$ (if $\Phi = \tilde{\Phi}$ and $\alpha = \tilde{\alpha}$) $\Rightarrow X \overset{d}{=} \tilde{X}$; but $X \overset{d}{=} \tilde{X} \not\Rightarrow X \overset{d}{=} \tilde{X}.$
- If $X \overset{d}{=} \tilde{X}$, then $\kappa = \tilde{\kappa}$ and $\alpha = \tilde{\alpha}$.
- Conversely, if $\kappa = \tilde{\kappa}$ and $\alpha = \tilde{\alpha}$, then $X \overset{d}{=} \tilde{X}$. [Pitman&Winkel, 2015; S. 2017]
Many-to-one formula

- Define the intensity measure $\mu^x_t$ of $X(t)$ on $\mathbb{R}_+$ such that for all $f \in C_c^\infty(\mathbb{R}_+)$:

$$\langle \mu^x_t, f \rangle := \int_{\mathbb{R}_+} f(y) \mu^x_t(dy) = \mathbb{E}_x \left[ \sum_{i=1}^{\infty} f(X_i(t)) \right].$$
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- $(\Phi^-(q) := \kappa(q + \omega_-), q \geq 0)$ is the Laplace exponent of a Lévy process $\eta^-$, and $\eta^-$ drifts to $-\infty$.

**Theorem (BBCK, 2016+)**

$$\mu^x_t(dy) = \left(\frac{x}{y}\right)^{\omega_-} \rho^-_t(x, dy), \quad y > 0,$$

where $\rho^-_t(x, \cdot)$ be the transition kernel of a pssMp $Y^-(t)$ with characteristics $(\Phi^-, \alpha)$ starting from $x > 0$. 

Quan Shi
Growth-Fragmentations
CIMAT, 11-15 December, 2017 28 / 34
Many-to-one formula

- Define the intensity measure $\mu^x_t$ of $X(t)$ on $\mathbb{R}_+$ such that for all $f \in C^\infty_c(\mathbb{R}_+)$:

$$\langle \mu^x_t, f \rangle := \int_{\mathbb{R}_+} f(y) \mu^x_t(dy) = E_x \left[ \sum_{i=1}^{\infty} f(X_i(t)) \right].$$

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$$\mu^x_t(dy) = \left( \frac{x}{y} \right)^{\omega_-} \rho^-_t(x, dy), \quad y > 0,$$

where $\rho^-_t(x, \cdot)$ be the transition kernel of a pssMp $Y^-(t)$ with characteristics $(\Phi^-, \alpha)$ starting from $x > 0$.

That is:

$$E_x \left[ \sum_{i=1}^{\infty} f(X_i(t)) \right] = E_x \left[ f(Y^-(t)) \left( \frac{x}{Y^-(t)} \right)^{\omega_-} 1_{\{Y^-(t) \in (0, \infty)\}} \right].$$
Proof of many-to-one formula

- For $\theta$ such that $\kappa(\theta) < 0$: $\kappa(\cdot + \theta)$ is the Laplace exponent of a Lévy process (with killing rate $-\kappa(\theta)$).
Proof of many-to-one formula

- For $\theta$ such that $\kappa(\theta) < 0$: $\kappa(\cdot + \theta)$ is the Laplace exponent of a Lévy process (with killing rate $-\kappa(\theta)$).

- Define

\[
\langle \tilde{\rho}_t(x, \cdot), f \rangle := x^{-\theta} E_x \left[ \sum_{i=1}^{\infty} f(X_i(t)) X_i(t)^{\theta} \right].
\]
Proof of many-to-one formula

- For $\theta$ such that $\kappa(\theta) < 0$: $\kappa(\cdot + \theta)$ is the Laplace exponent of a Lévy process (with killing rate $-\kappa(\theta)$).
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  \langle \tilde{\rho}_t(x, \cdot), f \rangle := x^{-\theta} E_x \left[ \sum_{i=1}^{\infty} f(X_i(t)) X_i(t)^\theta \right].
  \]
- \[
  \mu^x_t(dy) = \left( \frac{x}{y} \right)^\theta \tilde{\rho}_t(x, dy), \quad y > 0.
  \]
- $\tilde{\rho}_t$ is the transition kernel of a pssMp $\tilde{Y}$ with characteristics $(\kappa(\cdot + \theta), \alpha)$:
Proof of many-to-one formula

- For $\theta$ such that $\kappa(\theta) < 0$: $\kappa(\cdot + \theta)$ is the Laplace exponent of a Lévy process (with killing rate $-\kappa(\theta)$).

- Define
  \[
  \langle \tilde{\rho}_t(x, \cdot), f \rangle := x^{-\theta} \mathbb{E}_x \left[ \sum_{i=1}^{\infty} f(X_i(t)) X_i(t)^{\theta} \right].
  \]

- $\mu_t^x(dy) = \left( \frac{x}{y} \right)^{\theta} \tilde{\rho}_t(x, dy), \quad y > 0.$

- $\tilde{\rho}_t$ is the transition kernel of a pssMp $\tilde{Y}$ with characteristics $(\kappa(\cdot + \theta), \alpha)$:
  - $\langle \tilde{\rho}_{t+s}(x, \cdot), f \rangle = \int_{(0, \infty)} \langle \tilde{\rho}_t(y, \cdot), f \rangle \tilde{\rho}_s(x, dy)$ Chapman-Kolmogorov equation
  - $\langle \tilde{\rho}_t(x, \cdot), f \rangle = \langle \tilde{\rho}_{x^{\alpha}t}(1, \cdot), f(x \cdot) \rangle$ the self-similarity.
  - For every $q$ with $\kappa(q + \theta) < 0$:
    \[
    \int_0^\infty dt \int_{(0, \infty)} y^{q+\alpha} \tilde{\rho}_t(1, dy) = x^{-\theta} \mathbb{E}_x \left[ \int_0^\infty \left( \sum_{i=1}^{\infty} X_i(t)^{\theta+q+\alpha} \right) dt \right] = -\frac{1}{\kappa(\theta + q)}.
    \]
Growth-fragmentation equations

The intensity measure $\mu^x_t$ of $X(t)$: for all $f \in C_c^\infty(\mathbb{R}_+)$,

$$\langle \mu^x_t, f \rangle := \int_{\mathbb{R}_+} f(y) \mu^x_t(dy) = \mathbb{E}_x \left[ \sum_{i=1}^{\infty} f(X_i(t)) \right].$$

Proposition (Bertoin&Watson, 2016; BBCK, 2016+)

Suppose that $[H]$ holds. Then

$$\langle \mu^x_t, f \rangle = f(x) + \int_0^t \langle \mu^x_s, Gf \rangle \, ds,$$

where

$$Gf(y) := y^\alpha \left( \frac{1}{2} \sigma^2 f''(y) y^2 + \frac{c+1}{2} \sigma^2 f'(y) y + \int_{(-\infty,0)} (f(y e^z) + f(y(1-e^z)) - f(y)) \, \Lambda(\,dz) \right).$$

Proof: Use the many-to-one formula and the infinitesimal generator of the pssMp.
Growth-fragmentation equations

The intensity measure \( \mu^x_t \) of \( X(t) \): for all \( f \in C_c^\infty(\mathbb{R}_+) \),

\[
\langle \mu^x_t, f \rangle := \int_{\mathbb{R}_+} f(y) \mu^x_t(dy) = E \left[ \sum_{i=1}^\infty f(X_i(t)) \right].
\]

**Proposition (Bertoin\&Watson, 2016; BBCK, 2016+)**

*Suppose that \([H]\) holds. Then*

\[
\langle \mu^x_t, f \rangle = f(x) + \int_0^t \langle \mu^x_s, Gf \rangle ds,
\]

*where*

\[
Gf(y) := y^\alpha \left( \frac{1}{2} \sigma^2 f''(y)y^2 + (c + \frac{1}{2} \sigma^2)f'(y)y ight.
\]

\[
+ \int_{(-\infty,0)} \left( f(ye^z) + f(y(1 - e^z)) - f(y) + y(1 - e^z)f'(y) \right) \Lambda(dz) \right).
\]
Growth-fragmentation equations

The intensity measure $\mu_t^x$ of $X(t)$: for all $f \in C_c^\infty(\mathbb{R}_+)$,

$$\langle \mu_t^x, f \rangle := \int_{\mathbb{R}_+} f(y)\mu_t^x(dy) = \mathbb{E}_x \left[ \sum_{i=1}^{\infty} f(X_i(t)) \right].$$

**Proposition (Bertoin&Watson, 2016; BBCK, 2016+)**

Suppose that $[H]$ holds. Then

$$\langle \mu_t^x, f \rangle = f(x) + \int_0^t \langle \mu_s^x, Gf \rangle ds,$$

where

$$Gf(y) := y^\alpha \left( \frac{1}{2} \sigma^2 f''(y)y^2 + (c + \frac{1}{2} \sigma^2)f'(y)y \right.$$

$$+ \int_{(-\infty,0)} \left( f(ye^z) + f(y(1-e^z)) - f(y) + y(1-e^z)f'(y) \right) \Lambda(dz) \right).$$

**Proof:** Use the many-to-one formula and the infinitesimal generator of the pssMp $Y^-$. 
Temporal martingales

- Suppose that $[\mathcal{H}]$ holds. For the smaller root $\omega_-$ of $\kappa$, define

$$M^-(t) := \sum_{i=1}^{\infty} X_i(t)^{\omega_-}, \quad t \geq 0.$$ 

**Theorem (BBCK, 2016+)**

- When $\alpha \geq 0$, $M^-$ is a uniformly integrable $P_x$-martingale;
- When $\alpha < 0$, $M^-$ is a $P_x$-supermartingale which converges to 0 in $L^1(P_x)$.

**Proof:**

- By many-to-one formula: $E_x[M^-(t)] = x^{\omega_-} P(Y^-(t) \in (0, \infty))$;
- If $\alpha \geq 0$, then $P(Y^-(t) \in (0, \infty)) \equiv 1$;
- If $\alpha < 0$, then $\lim_{t \to \infty} P(Y^-(t) \in (0, \infty)) = 0$;
- We conclude by the branching property of $X$. 
Temporal martingales

Suppose that \([H]\) holds. For the larger root \(\omega_+\) of \(\kappa\), define

\[
M^+(t) := \sum_{i=1}^{\infty} X_i(t)^{\omega_+}, \quad t \geq 0.
\]

Theorem (BBCK, 2016+)

- When \(\alpha > 0\), \(M^+\) is a \(P_x\)-supermartingale which converges to 0 in \(L^1(P_x)\);
- When \(\alpha \leq 0\), \(M^+\) is a \(P_x\)-martingale.
Summary

- $\text{pssMp } X \Rightarrow \text{ cell system } (\mathcal{X}_u, u \in \mathbb{U}) \Rightarrow \text{ growth-fragmentation } X$
- The law of $X$ is characterized by the cumulant $\kappa$ and the index of self-similarity $\alpha$
- Non-explosion condition: there exists $\kappa(q) \leq 0$.
- $X$ satisfies the branching property and the self-similarity.
- The many-to-one formula
- The intensity measure of $X$ solves a (deterministic) growth-fragmentation equation
- If $\kappa(\omega^-) = \kappa(\omega^+) = 0$: two martingales $M^-$ and $M^+$
Thank you!