# The hitting time of zero for stable processes

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## Stable process

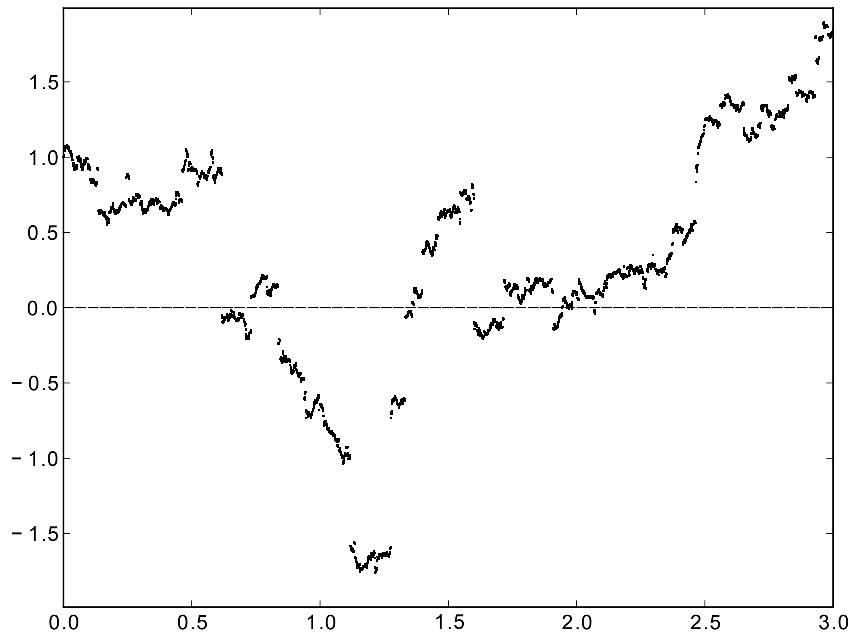
Begin with a stable process, $X$:

- Lévy process satisfying the scaling property

$$(cX_{tc^{-\alpha}})_{t\geq 0} \stackrel{d}{=} X, \qquad c > 0.$$

- Characterised by two parameters, $\alpha$ and $\rho$: $\rho = \mathrm{P}(X_t > 0)$.
- Our focus: $\alpha \in (1, 2)$.

## Stable process: sample path, $(\alpha, \rho) = (1.3, 0)$






## Problem: statement

Let

$$T_0 = \inf\{t \geq 0 : X_t = 0\}.$$

The problem: characterise $\mathrm{P}_x(T_0 \in \cdot)$

We will find:

- The Mellin transform, $\mathrm{E}_x[T_0^{s-1}]$
- The density, $\mathrm{P}_x(T_0 \in \mathrm{d}y)/\mathrm{d}y$

## Problem: history

- Spectrally one-sided case: Peskir (2008)
- Symmetric case: Yano, Yano, Yor (2009) and Cordero (2010)



## Real self-similar Markov processes

**$\alpha$-rssMp**

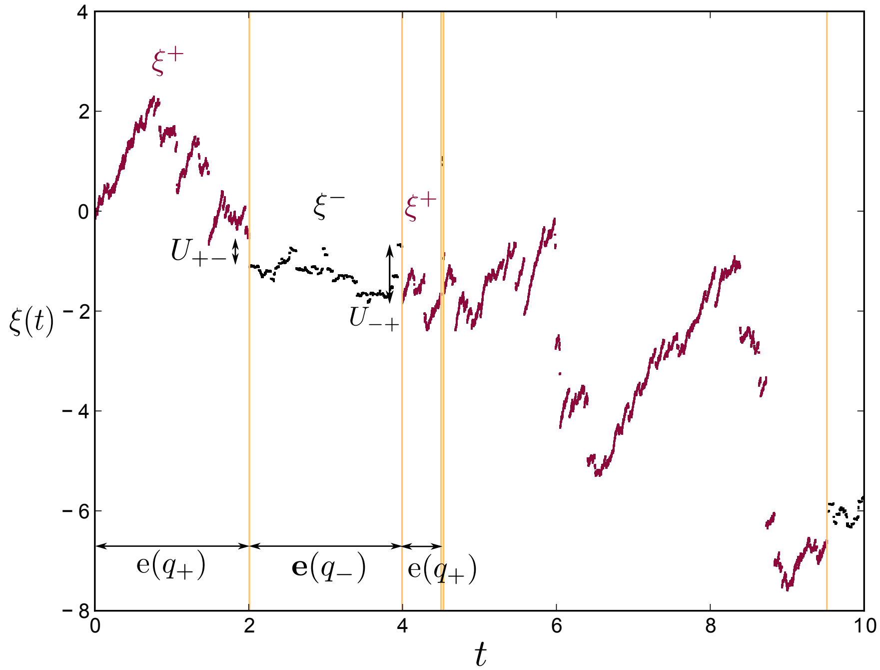
- $\mathbb{R}$-valued Markov process
- with initial measures $\mathrm{P}_x$, $x \neq 0$
- 0 an absorbing state
- satisfying the scaling property

$$(cX_{tc^{-\alpha}})_{t \geq 0}|_{\mathrm{P}_x} \stackrel{d}{=} X|_{\mathrm{P}_{cx}}, \qquad x \neq 0,\ c > 0$$

Assume $X$ jumps over zero in both directions

## Two state $(+,-)$ Markov additve process

Lévy processes $\xi^+$,$\xi^-$; clock rates $q_+$,$q_-$; jumps $U_{+-}$,$U_{-+}$
Markov additive process $(\xi(t), J(t))_{t\geq 0}$






## MAP: characterisation

There exists a matrix $F(z) = \begin{pmatrix} F_{++}(z) & F_{+-}(z) \\ F_{-+}(z) & F_{--}(z) \end{pmatrix}$ such that

$$\left(e^{tF(z)}\right)_{ij} = \mathbb{E}\left[e^{z\xi(t)}; J(t) = j \mid J(0) = i, \, \xi(0) = 0\right].$$

$F$ is determined by the components:

$$F(z) = \begin{pmatrix} \psi_+(z) & 0 \\ 0 & \psi_-(z) \end{pmatrix} + \begin{pmatrix} -q_+ & q_+ G_{+-}(z) \\ q_- G_{-+}(z) & -q_- \end{pmatrix}$$

where $e^{\psi_i(z)} = \mathbb{E}[e^{z\xi^i(1)}]$ and $G_{ij}(z) = \mathbb{E}[e^{zU_{ij}}]$.

## Lamperti–Kiu transform (Chaumont, Pantí, Rivero)

| $\alpha$-rssMp $X$ under $P_{\pm 1}$ | Two-state MAP $(\xi, J)$ |
| --- | --- |
| $X_t = J(S(t)) \cdot \exp(\xi(S(t))$ | $\xi(s) = \log(\lvert X_{T(s)}\rvert)$ <br> $J(s) = \mathrm{sgn}(X_{T(s)})$ |
| $S(t)$ inverse to $\int_0^\cdot e^{\alpha\xi(u)}\,\mathrm{d}u$ | $T(s)$ inverse to $\int_0^\cdot \lvert X_u\rvert^{-\alpha}\,\mathrm{d}u$ |
| $X$ never hits zero <br> $X$ hits zero continuously | $\xi \to \infty$ or $\xi$ oscillates <br> $\xi \to -\infty$ |
| $T_0 = T(\infty) = \int_0^\infty e^{\alpha\xi(u)}\,\mathrm{d}u$ | |



## Lamperti–Kiu transform: example

$X$: stable process killed on hitting zero (an $\alpha$-rssMp)

Compute explicitly

$$F(z) = \frac{\Gamma(\alpha - z)\Gamma(1+z)}{\pi} \begin{pmatrix} -\sin(\pi(\alpha\hat{\rho} - z)) & \sin(\pi\alpha\hat{\rho}) \\ \sin(\pi\alpha\rho) & -\sin(\pi(\alpha\rho - z)) \end{pmatrix}$$

# Exponential functional

Recall:

$$T_0 = I(\alpha\xi) := \int_0^\infty e^{\alpha\xi(u)}\,\mathrm{d}u.$$

How to characterise the exponential functional of $\xi$?
Let

$$\mathcal{M}(s) = \begin{pmatrix} \mathcal{M}_+(s) \\ \mathcal{M}_-(s) \end{pmatrix} = \begin{pmatrix} \mathbb{E}[I(\alpha\xi)^{s-1} \mid J(0) = +] \\ \mathbb{E}[I(\alpha\xi)^{s-1} \mid J(0) = -] \end{pmatrix},$$

the Mellin transform of $I(\alpha\xi)$.



# Exponential functional

**Proposition**

*Provided $(\xi, J)$ satisfies a Cramér condition,*

$$\mathcal{M}(s+1) = -s(F(\alpha s))^{-1}\mathcal{M}(s),$$

*for certain $s \in \mathbb{R}$.*

## Solution: Mellin transform

**Theorem**

*For* $\operatorname{Re} s \in (-1/\alpha, 2 - 1/\alpha)$,

$$\mathrm{E}_1[T_0^{s-1}] = \frac{\sin\left(\frac{\pi}{\alpha}\right)}{\sin(\pi\hat{\rho})} \frac{\sin\left(\pi\hat{\rho}(1-\alpha+\alpha s)\right)}{\sin\left(\frac{\pi}{\alpha}(1-\alpha+\alpha s)\right)} \frac{\Gamma(1+\alpha-\alpha s)}{\Gamma(2-s)}.$$



## Solution: density

Let $\mathrm{P}_1(T_0 \in \mathrm{d}t) = p(t)\,\mathrm{d}t$.

**Theorem**

*For a dense, full-measure set of* $\alpha \in (1,2)$*:*

$$p(t) = \frac{\sin\left(\frac{\pi}{\alpha}\right)}{\pi\sin(\pi\hat{\rho})} \sum_{k\geq 1} \frac{\sin(\pi\hat{\rho}(k+1))\sin(\frac{\pi}{\alpha}k)}{\sin(\frac{\pi}{\alpha}(k+1))} \frac{\Gamma(\frac{k}{\alpha}+1)}{k!}(-1)^{k-1}t^{-1-\frac{k}{\alpha}}$$
$$- \frac{\sin\left(\frac{\pi}{\alpha}\right)^2}{\pi\sin(\pi\hat{\rho})} \sum_{k\geq 1} \frac{\sin(\pi\alpha\hat{\rho}k)}{\sin(\pi\alpha k)} \frac{\Gamma\left(k-\frac{1}{\alpha}\right)}{\Gamma(\alpha k - 1)} t^{-k-1+\frac{1}{\alpha}}.$$

## Further reading

A. Kuznetsov, A. E. Kyprianou, J. C. Pardo, A. R. Watson
The hitting time of zero for a stable process
arXiv:1212.5153 [math.PR]



# Thank you!