The hitting time of zero for stable processes

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Stable process

Begin with a stable process, X:

- Lévy process satisfying the scaling property

$$(cX_{tc^{-\alpha}})_{t\geq 0} \stackrel{d}{=} X, \qquad c > 0.$$

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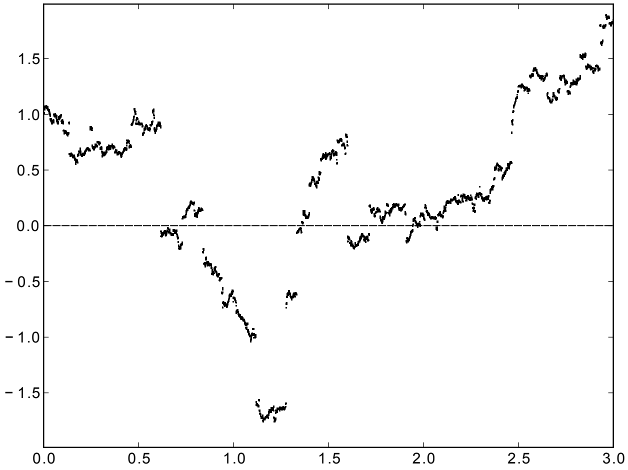
- Characterised by two parameters, α and ρ:
  $\rho = \mathrm{P}(X_t > 0)$.
- Our focus: $\alpha \in (1, 2)$.

Stable process: sample path, $(\alpha, \rho) = (1.3, 0)$



Problem: statement

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We will find:

- The Mellin transform, $E_x[T_0^{s-1}]$
- The density, $P_x(T_0 \in dy)/dy$

Problem: history

- Spectrally one-sided case: Peskir (2008)
- Symmetric case: Yano, Yano, Yor (2009) and Cordero (2010)

Real self-similar Markov processes

α-rssMp

- $\mathbb{R}$-valued Markov process
- with initial measures P_x, $x \neq 0$
- 0 an absorbing state
- satisfying the scaling property

$$(cX_{tc^{-\alpha}})_{t\geq 0}|_{\mathrm{P}_x} \overset{d}{=} X|_{\mathrm{P}_{cx}}, \qquad x \neq 0, \; c > 0$$

Assume X jumps over zero in both directions

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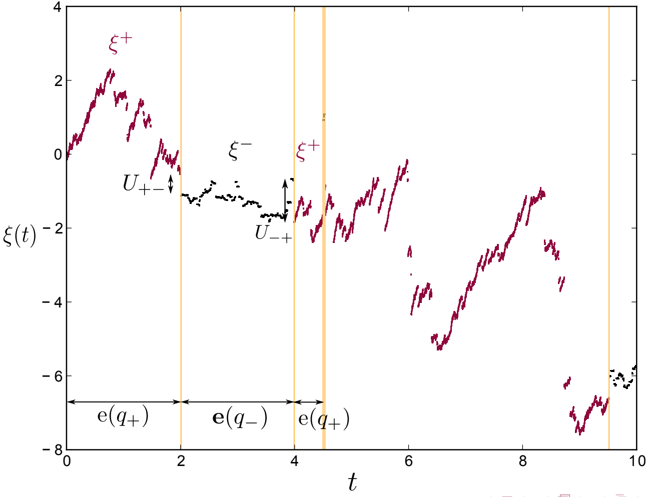
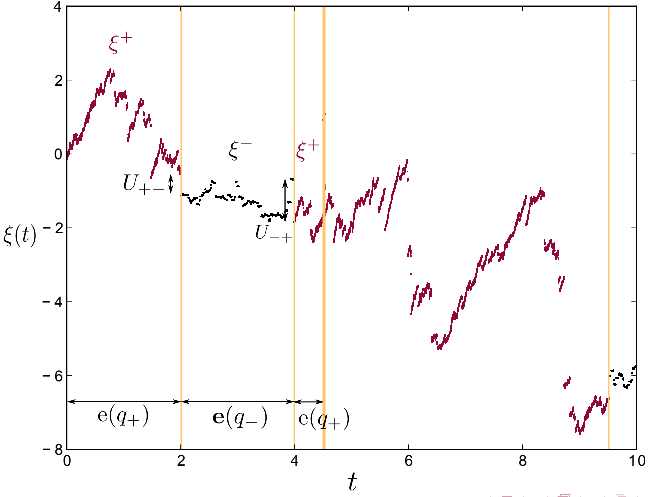
Two state $(+,-)$ Markov additve process

Lévy processes ξ^+,ξ^-; clock rates q_+,q_-; jumps U_{+-},U_{-+}



Two state $(+,-)$ Markov additve process

Lévy processes ξ^+,ξ^-; clock rates q_+,q_-; jumps U_{+-},U_{-+}
Markov additive process $(\xi(t), J(t))_{t\geq 0}$



MAP: characterisation

There exists a matrix $F(z) = \begin{pmatrix} F_{++}(z) & F_{+-}(z) \\ F_{-+}(z) & F_{--}(z) \end{pmatrix}$ such that

$$\left(e^{tF(z)}\right)_{ij} = \mathbb{E}\left[e^{z\xi(t)}; J(t) = j \mid J(0) = i,\, \xi(0) = 0\right].$$

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F is determined by the components:

$$F(z) = \begin{pmatrix} \psi_+(z) & 0 \\ 0 & \psi_-(z) \end{pmatrix} + \begin{pmatrix} -q_+ & q_+ G_{+-}(z) \\ q_- G_{-+}(z) & -q_- \end{pmatrix}$$

where $e^{\psi_i(z)} = \mathbb{E}[e^{z\xi^i(1)}]$ and $G_{ij}(z) = \mathbb{E}[e^{zU_{ij}}]$.

Lamperti–Kiu transform (Chaumont, Pantí, Rivero)

α-rssMp X under $\mathrm{P}_{\pm 1}$

$X_t = J(S(t)) \cdot \exp(\xi(S(t))$

$S(t)$ inverse to $\int_0^\cdot e^{\alpha\xi(u)}\,\mathrm{d}u$

Two-state MAP (ξ, J)

$\xi(s) = \log(|X_{T(s)}|)$

$J(s) = \mathrm{sgn}(X_{T(s)})$

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$T_0 = T(\infty) = \int_0^{\infty} e^{\alpha \xi(u)}\,\mathrm{d}u$

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Lamperti–Kiu transform: example

X: stable process killed on hitting zero (an α-rssMp)

Compute explicitly

$$F(z) = \frac{\Gamma(\alpha - z)\Gamma(1+z)}{\pi} \begin{pmatrix} -\sin(\pi(\alpha\hat{\rho} - z)) & \sin(\pi\alpha\hat{\rho}) \\ \sin(\pi\alpha\rho) & -\sin(\pi(\alpha\rho - z)) \end{pmatrix}$$

Exponential functional

Recall:

$$T_0 = I(\alpha\xi) := \int_0^\infty e^{\alpha\xi(u)}\,\mathrm{d}u.$$

How to characterise the exponential functional of ξ?

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Let

$$\mathcal{M}(s) = \begin{pmatrix} \mathcal{M}_+(s) \\ \mathcal{M}_-(s) \end{pmatrix} = \begin{pmatrix} \mathbb{E}[I(\alpha\xi)^{s-1} \mid J(0) = +] \\ \mathbb{E}[I(\alpha\xi)^{s-1} \mid J(0) = -] \end{pmatrix},$$

the Mellin transform of $I(\alpha\xi)$.

Exponential functional

Proposition

Provided (ξ, J) satisfies a Cramér condition,

$$\mathcal{M}(s+1) = -s(F(\alpha s))^{-1}\mathcal{M}(s),$$

for certain $s \in \mathbb{R}$.

Solution: Mellin transform

Theorem

For $\operatorname{Re} s \in (-1/\alpha, 2 - 1/\alpha)$,

$$\mathrm{E}_1[T_0^{s-1}] = \frac{\sin\left(\frac{\pi}{\alpha}\right)}{\sin(\pi\hat{\rho})} \frac{\sin\left(\pi\hat{\rho}(1-\alpha+\alpha s)\right)}{\sin\left(\frac{\pi}{\alpha}(1-\alpha+\alpha s)\right)} \frac{\Gamma(1+\alpha-\alpha s)}{\Gamma(2-s)}.$$

Solution: density

Let $\mathrm{P}_1(T_0 \in \mathrm{d}t) = p(t)\,\mathrm{d}t$.

Theorem

For a dense, full-measure set of $\alpha \in (1,2)$*:*

$$p(t) = \frac{\sin\left(\frac{\pi}{\alpha}\right)}{\pi \sin(\pi\hat{\rho})} \sum_{k\geq 1} \frac{\sin(\pi\hat{\rho}(k+1))\sin(\frac{\pi}{\alpha}k)}{\sin(\frac{\pi}{\alpha}(k+1))} \frac{\Gamma(\frac{k}{\alpha}+1)}{k!}(-1)^{k-1} t^{-1-\frac{k}{\alpha}}$$

$$- \frac{\sin\left(\frac{\pi}{\alpha}\right)^2}{\pi \sin(\pi\hat{\rho})} \sum_{k\geq 1} \frac{\sin(\pi\alpha\hat{\rho}k)}{\sin(\pi\alpha k)} \frac{\Gamma\left(k-\frac{1}{\alpha}\right)}{\Gamma(\alpha k - 1)} t^{-k-1+\frac{1}{\alpha}}.$$

Further reading

A. Kuznetsov, A. E. Kyprianou, J. C. Pardo, A. R. Watson
The hitting time of zero for a stable process
arXiv:1212.5153 [math.PR]

Thank you!