The hitting time of zero for stable processes

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Abstract. We will discuss a method to characterise explicitly the law of the first time at which a stable process reaches the point zero, using theories of self-similar Markov processes. When the process is symmetric, the Lamperti representation characterises the law of the hitting time of zero as equal to that of the exponential functional of a Lévy process. When the stable process is asymmetric, things are not quite so simple. However, using the newly-developed theory of real self-similar Markov processes, we demonstrate that the hitting time of zero is equal to the exponential functional of a Markov additive process. These laws have not been investigated very thoroughly in the past, but remarkably, we are able to set up and solve a two-dimensional functional equation for a vector-valued Mellin transform. Moreover, we can even write down the density of the hitting time. Finally, we will discuss an application to the stable process conditioned to avoid zero.

Based on [4], which is joint work with Alexey Kuznetsov (York, Canada), Juan-Carlos Pardo (CIMAT) and Andreas Kyprianou (Bath, UK).

About these notes. This document is generated from my `beamer` presentation based on notes left in the source. My hope is that the reader who wishes they had attended my talk, but found it necessary instead to file some expenses/write a grant proposal/play Minecraft, will be able to read this and obtain a similar experience in the comfort of their own home. It should read somewhere between an informal talk and a formal article, and as such there are likely to be some inaccuracies due to the informal language and the fact that I wrote most of it from memory.

This document is probably best read along with the accompanying slides, but 'flattened' (and somewhat mangled) versions are included for convenience, demarcated by horizonal rules.

Are you sitting comfortably? Then I'll begin.

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To state the problem, we will begin with the stable process. This is simple a Lévy process – a process with stationary, independent increments – which also satisfies a certain property of scaling invariance. Stable processes in one dimension are characterised by two parameters, a *scaling parameter* $\alpha \in (1, 2]$; and a *positivity parameter* ρ. In general, the range of parameters is

$$\{\alpha \in (0, 1], \rho \in [0, 1]\} \cup \{\alpha \in (1, 2), \rho \in [1 - 1/\alpha, 1/\alpha]\} \cup \{(\alpha, \rho) = (2, 1/2)\}.$$

However, for reasons which will soon become apparent, we will focus on the case where $\alpha \in (1, 2)$; indeed, we will only deal with the parameter set

$$\{\alpha \in (1, 2), \rho \in (1 - 1/\alpha, 1/\alpha)\}.$$

Stable process

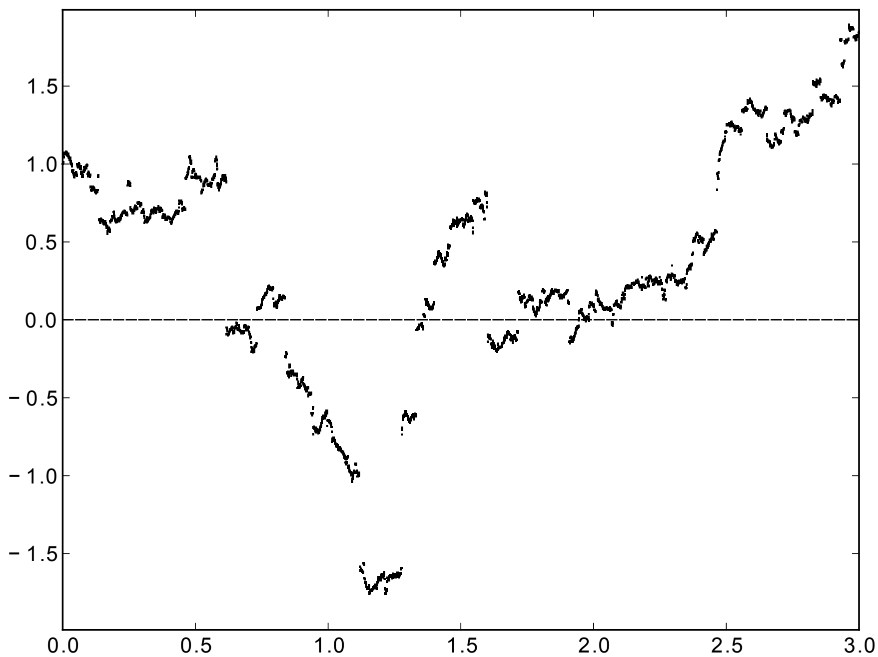
Begin with a stable process, X:

- Lévy process satisfying the *scaling property*

$$(cX_{tc^{-\alpha}})_{t\geq 0} \stackrel{d}{=} X, \qquad c > 0.$$

- Characterised by two parameters, α and ρ: $\rho = \mathrm{P}(X_t > 0)$.
- Our focus: $\alpha \in (1, 2)$.

Stable process: sample path, $(\alpha, \rho) = (1.3, 0)$



The big problem is to characterise the law of the first hitting time of zero. We will do it in two ways, firstly by computing the Mellin transform, and then by inverting that transform to give expressions for the density of the law.

Problem: statement

Let

$$T_0 = \inf\{t \geq 0 : X_t = 0\}.$$

The problem: *characterise* $\mathrm{P}_x(T_0 \in \cdot)$

We will find:

- The *Mellin transform*, $\mathrm{E}_x[T_0^{s-1}]$
- The *density*, $\mathrm{P}_x(T_0 \in \mathrm{d}y)/\mathrm{d}y$

The problem has been studied previously. The more difficult spectrally one-sided case is given in [5], and the easier case may be deduced via the theory of scale functions. The symmetric case is considered in [6], where T_0 is characterised as the product of two laws; and [3], where an explicit expression for the Mellin transform is given.

However, these works rely heavily on potential theory. Our method is somewhat different, in that we make use of a path transformation.

Problem: history

- Spectrally one-sided case: Peskir (2008)
- Symmetric case: Yano, Yano, Yor (2009) and Cordero (2010)

Our insight into the problem will come by viewing the stable process killed on hitting zero as a *real self-similar Markov process*. The theory behind these processes was developed by Chaumont et al. [1]. Note that, in the fourth bullet point of the slide, it is crucial that $c > 0$; this allows the process to have, essentially, two different behaviours depending on whether it is positive or negative.

We are going to make the assumption that X jumps both from positive to negative and from negative to positive in finite time. The processes where this does not occur have a similar structure but get 'stuck' in either a positive or negative state; it is simpler for us to omit them, but the majority of what we will discuss holds also for these processes.

Real self-similar Markov processes

α-rssMp

- $\mathbb{R}$-valued Markov process
- with initial measures P_x, $x \neq 0$
- *0 an absorbing state*
- satisfying the *scaling property*

$$(cX_{tc^{-\alpha}})_{t\geq 0}|_{\mathrm{P}_x} \stackrel{d}{=} X|_{\mathrm{P}_{cx}}, \qquad x \neq 0,\ c > 0$$

Assume X jumps over zero in both directions

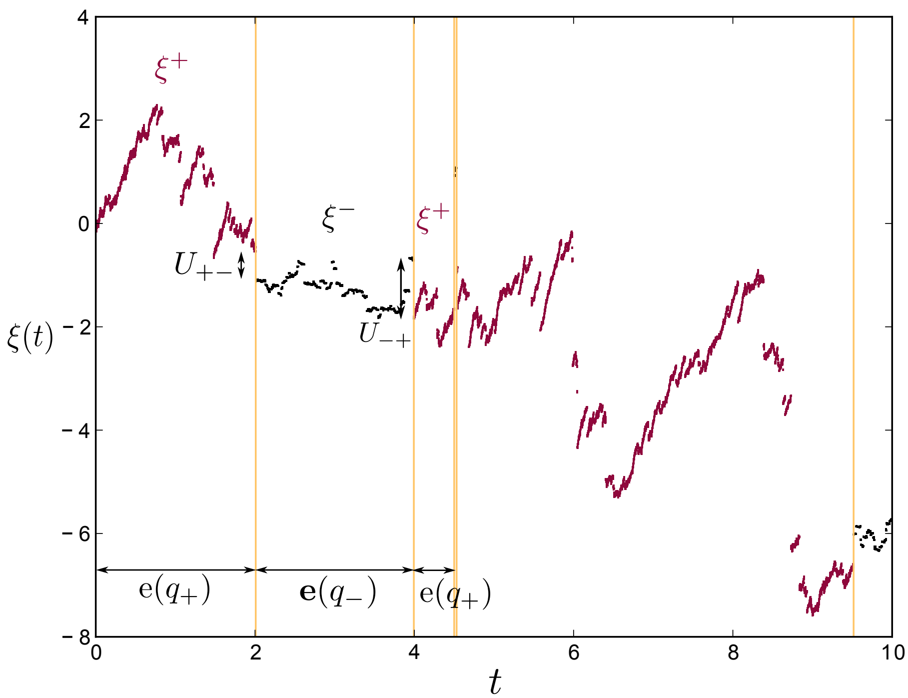
To describe the Lamperti–Kiu transform, we need to talk about Markov additive processes. For the sake of simplicity, we will just consider two-state MAPs; the full theory for finite-state MAPs is very similar, but the notation is more irritating. We will call the two states $+$ and $-$, for reasons which will be revealed in two slides' time.

A Markov additive process (ξ, J) runs as follows. Suppose that J begins in the state $+$. Then ξ follows a Lévy process with law ξ^+, and runs until a independent exponentially-distributed clock of rate q_+ rings; then, ξ makes a(n independent) jump distributed as U_{+-}, the process J switches to state $-$, and everything starts afresh, with ξ following a Lévy process ξ^- instead.

Two state $(+,-)$ Markov additve process

Lévy processes ξ^+,ξ^-; clock rates q_+,q_-; jumps U_{+-},U_{-+}

Markov additive process $(\xi(t), J(t))_{t\geq 0}$



Much as a Lévy process is characterised by its characteristic exponent, a MAP is characterised by a matrix, given here. The fact that it can be written fairly simply in terms of the components of the MAP will allow us to compute specific examples shortly.

MAP: characterisation

There exists a matrix $F(z) = \begin{pmatrix} F_{++}(z) & F_{+-}(z) \\ F_{-+}(z) & F_{--}(z) \end{pmatrix}$ such that

$$\left(e^{tF(z)}\right)_{ij} = \mathbb{E}\left[e^{z\xi(t)}; J(t) = j \mid J(0) = i,\, \xi(0) = 0\right].$$

F is determined by the components:

$$F(z) = \begin{pmatrix} \psi_+(z) & 0 \\ 0 & \psi_-(z) \end{pmatrix} + \begin{pmatrix} -q_+ & q_+ G_{+-}(z) \\ q_- G_{-+}(z) & -q_- \end{pmatrix}$$

where $e^{\psi_i(z)} = \mathbb{E}[e^{z\xi^i(1)}]$ and $G_{ij}(z) = \mathbb{E}[e^{zU_{ij}}]$.

The Lamperti–Kiu transform, given by Chybiryakov [2] in the symmetric case and in general by Chaumont et al. [1], gives a bijection between rssMps and two-state Markov additive processes. The transform is essentially the same as the Lamperti transform (between positive self-similar Markov processes and Lévy processes) but it uses the Markov chain component of the Markov additive process to keep track of the sign of X.

We remark on the long-term behaviour of the processes X and ξ; in particular, when X hits zero continuously, the hitting time is given by an *exponential functional* of ξ.

Finally, the slide gives the results when starting the rssMp from ± 1. Since X satisfies a scaling property, it is not difficult to deduce from this the case where it starts from $x \neq 0$, but we omit the details.

Lamperti–Kiu transform (Chaumont, Pantí, Rivero)

α-rssMp X under $\mathrm{P}_{\pm 1}$

$X_t = J(S(t)) \cdot \exp(\xi(S(t))$

$S(t)$ inverse to $\int_0^\cdot e^{\alpha\xi(u)}\,\mathrm{d}u$

$$\left.\begin{array}{r} X \text{ never hits zero} \\ X \text{ hits zero continuously} \end{array}\right\}$$

$T_0 = T(\infty) = \int_0^\infty e^{\alpha\xi(u)}\,\mathrm{d}u$

Two-state MAP (ξ, J)

$\xi(s) = \log(|X_{T(s)}|)$

$J(s) = \mathrm{sgn}(X_{T(s)})$

$T(s)$ inverse to $\int_0^\cdot |X_u|^{-\alpha}\,\mathrm{d}u$

$$\left\{\begin{array}{c} \xi \to \infty \text{ or } \xi \text{ oscillates} \\ \xi \to -\infty \end{array}\right.$$

Before we get to the main topic of the talk, let's see an example – it is one of the few examples where we can compute the Lamperti–Kiu transform explicitly.

The components of this MAP were computed in [1]. We remark here that the diagonal entries in F are the Laplace exponents of hypergeometric, Lamperti-stable Lévy processes; and the off-diagonal entries correspond to 'log-Pareto' distributions.

Lamperti–Kiu transform: example

X: stable process killed on hitting zero (an α-rssMp)

Compute explicitly

$$F(z) = \frac{\Gamma(\alpha - z)\Gamma(1+z)}{\pi} \begin{pmatrix} -\sin(\pi(\alpha\hat{\rho} - z)) & \sin(\pi\alpha\hat{\rho}) \\ \sin(\pi\alpha\rho) & -\sin(\pi(\alpha\rho - z)) \end{pmatrix}$$

Exponential functional

Recall:

$$T_0 = I(\alpha\xi) := \int_0^\infty e^{\alpha\xi(u)}\,\mathrm{d}u.$$

How to characterise the *exponential functional* of ξ?

Let

$$\mathcal{M}(s) = \begin{pmatrix} \mathcal{M}_+(s) \\ \mathcal{M}_-(s) \end{pmatrix} = \begin{pmatrix} \mathbb{E}[I(\alpha\xi)^{s-1} \mid J(0) = +] \\ \mathbb{E}[I(\alpha\xi)^{s-1} \mid J(0) = -] \end{pmatrix},$$

the *Mellin transform* of $I(\alpha\xi)$.

The principal ingredient in finding $\mathcal{M}$ is the following vector-valued functional equation, which generalises the well-known scalar equation for the exponential functional of a Lévy process. It holds whenever the Markov additive process satisfies a *Cramér condition*, which we make precise here.

Whenever the matrix $F(z)$ is defined, it is known, by Perron–Frobenius theory, to have a simple real eigenvalue with minimal real part; denote this by $k(z)$. This leading eigenvalue, so called, may be viewed as an equivalent of the Laplace exponent of a Lévy process, and it has some similar properties; for instance, it is smooth and convex wherever it is defined.

We say that (ξ, J) satisfies the Cramér with Cramér number $\theta > 0$ if k is defined on $(0, \theta+\epsilon)$, for some $\epsilon > 0$, and $k(\theta) = 0$. This is a sufficient condition for $\mathcal{M}$ to be finite for all $\operatorname{Re} s \in (0, 1+\theta)$ and for the functional equation in the slide to hold whenever $s \in (0, \theta)$.

Exponential functional

Proposition 1. *Provided (ξ, J) satisfies a Cramér condition,*

$$\mathcal{M}(s+1) = -s(F(\alpha s))^{-1}\mathcal{M}(s),$$

for certain $s \in \mathbb{R}$.

We have a good intuition about the form of the Mellin transform due to the known cases mentioned, and when combined with the functional equation above, this is enough to deduce the correct solution. Verifying this is not easy, but the basic outline is much as in the case of Lévy processes, and consists of complex analytic arguments using the asymptotics of the proposed solution.

Solution: Mellin transform

Theorem 2. *For* $\operatorname{Re} s \in (-1/\alpha, 2-1/\alpha)$,

$$\mathrm{E}_1[T_0^{s-1}] = \frac{\sin\left(\frac{\pi}{\alpha}\right)}{\sin(\pi\hat{\rho})} \frac{\sin\left(\pi\hat{\rho}(1-\alpha+\alpha s)\right)}{\sin\left(\frac{\pi}{\alpha}(1-\alpha+\alpha s)\right)} \frac{\Gamma(1+\alpha-\alpha s)}{\Gamma(2-s)}.$$

Computing the density is in principle rather a simple matter, involving simply the inversion integral for Mellin transforms and residue calculus. In practice, it turns out to be somewhat more involved, and the delicate bounds required to ensure convergence yield the following (partial) result; more on the 'dense set', and what to do outside this set, can be found in the article.

Solution: density

Let $\mathrm{P}_1(T_0 \in \mathrm{d}t) = p(t)\,\mathrm{d}t$.
Theorem 3. *For a dense, full-measure set of $\alpha \in (1,2)$:*

$$p(t) = \frac{\sin\left(\frac{\pi}{\alpha}\right)}{\pi \sin(\pi\hat{\rho})} \sum_{k\geq 1} \frac{\sin(\pi\hat{\rho}(k+1))\sin(\frac{\pi}{\alpha}k)}{\sin(\frac{\pi}{\alpha}(k+1))} \frac{\Gamma(\frac{k}{\alpha}+1)}{k!}(-1)^{k-1} t^{-1-\frac{k}{\alpha}}$$
$$- \frac{\sin\left(\frac{\pi}{\alpha}\right)^2}{\pi \sin(\pi\hat{\rho})} \sum_{k\geq 1} \frac{\sin(\pi\alpha\hat{\rho}k)}{\sin(\pi\alpha k)} \frac{\Gamma\left(k-\frac{1}{\alpha}\right)}{\Gamma(\alpha k - 1)} t^{-k-1+\frac{1}{\alpha}}.$$

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