The hitting time of zero for stable processes, symmetric and asymmetric

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Stable process

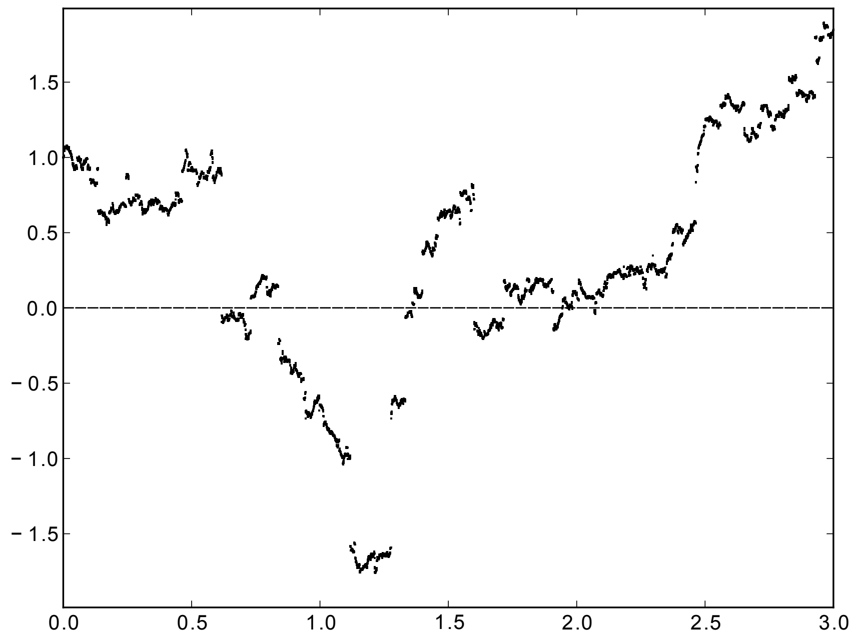
Begin with a stable process, X:

- Lévy process satisfying the scaling property

$$(cX_{tc^{-\alpha}})_{t\geq 0} \stackrel{d}{=} X, \qquad c > 0.$$

- Characterised by two parameters, α and ρ: $\rho = \mathrm{P}(X_t > 0)$.
- Our focus: $\alpha \in (1, 2)$.

Stable process: sample path, $(\alpha, \rho) = (1.3, 0)$



Problem: statement

Let

$$T_0 = \inf\{t \geq 0 : X_t = 0\}.$$

The problem: characterise $\mathrm{P}_x(T_0 \in \cdot)$, $\quad x \neq 0$

We will find:

- The Mellin transform, $\mathrm{E}_x[T_0^{s-1}]$
- The density, $\mathrm{P}_x(T_0 \in \mathrm{d}y)/\mathrm{d}y$

Problem: history

- Spectrally one-sided case: Peskir (2008) and Simon (2011)
- Symmetric case: Yano, Yano, Yor (2009) and Cordero (2010)

Positive self-similar Markov processes

α-pssMp

- $[0,\infty)$-valued Markov process
- with initial measures P_x, $x > 0$
- 0 an absorbing state
- satisfying the scaling property

$$(cX_{tc^{-\alpha}})_{t\geq 0}|_{\mathrm{P}_x} \stackrel{d}{=} X|_{\mathrm{P}_{cx}}, \qquad x \neq 0,\ c > 0$$

pssMps: examples

- Brownian motion killed on hitting zero
- Bessel process (of any dimension) killed on hitting zero
- Stable process...
  - ...killed on exiting $(0, \infty)$
  - ...conditioned to remain in $(0, \infty)$
  - ...conditioned to hit 0 continuously from above
  - ...censored outside of $(0, \infty)$

Finally, if X is a symmetric stable process, then $|X|$ is a pssMp, and if $\alpha > 1$, it hits zero.

Lamperti transform

α-pssMp X	(killed) Lévy process ξ
$X_t = \exp(\xi_{S(t)})$	$\xi_s = \log(X_{T(s)})$
$S(t)$ inverse to $\int_0^{\cdot} e^{\alpha\xi_u}\,du$	$T(s)$ inverse to $\int_0^{\cdot}(X_u)^{-\alpha}\,du$
X never hits zero	$\xi \to \infty$ or ξ oscillates
X hits zero continuously	$\xi \to -\infty$
X hits zero by a jump	ξ is killed
$T_0 = T(\infty) = \int_0^{\infty} e^{\alpha\xi_u}\,du$	

Apply to $|X|$

Write ξ for the Lamperti transform.

- Laplace exponent, $\mathbb{E}[e^{\lambda X_t}] = e^{t\psi(\lambda)}$:

$$\psi(\lambda) = -2^\alpha \frac{\Gamma(\alpha/2 - \lambda/2)}{\Gamma(-\lambda/2)} \frac{\Gamma(1/2 + \lambda/2)}{\Gamma((1-\alpha)/2 + \lambda/2)},$$

  for $\operatorname{Re}\lambda \in (-1, 1/\alpha)$
- hypergeometric / extended hypergeometric class

Exponential functional

$$T_0 = I := \int_0^\infty e^{\alpha \xi_u} \, du.$$

Let ξ be a Lévy process satisfying a Cramér condition, i.e. for some $\theta > 0$, $\psi(\theta) = 0$.

Write

$$\mathcal{M}(s) = \mathbb{E}[I^{s-1}].$$

Then

$$\mathcal{M}(s+1) = -\frac{s}{\psi(\alpha s)} \mathcal{M}(s),$$

and this, together with an asymptotic condition, is often enough to find $\mathcal{M}$.

Main result, symmetric case

$$\mathrm{E}_1[T_0^{s-1}] = \hat{\mathbb{E}}_0[I^{s-1}] = \mathcal{M}(s)$$

Proposition

For $\mathrm{Re}\, s \in (-1/\alpha, 2 - 1/\alpha)$,

$$\mathcal{M}(s) = \sin(\pi/\alpha) \frac{\cos(\frac{\pi\alpha}{2}(s-1))}{\sin(\pi(s-1+\frac{1}{\alpha}))} \frac{\Gamma(1+\alpha-\alpha s)}{\Gamma(2-s)}$$

Real self-similar Markov processes

α-rssMp

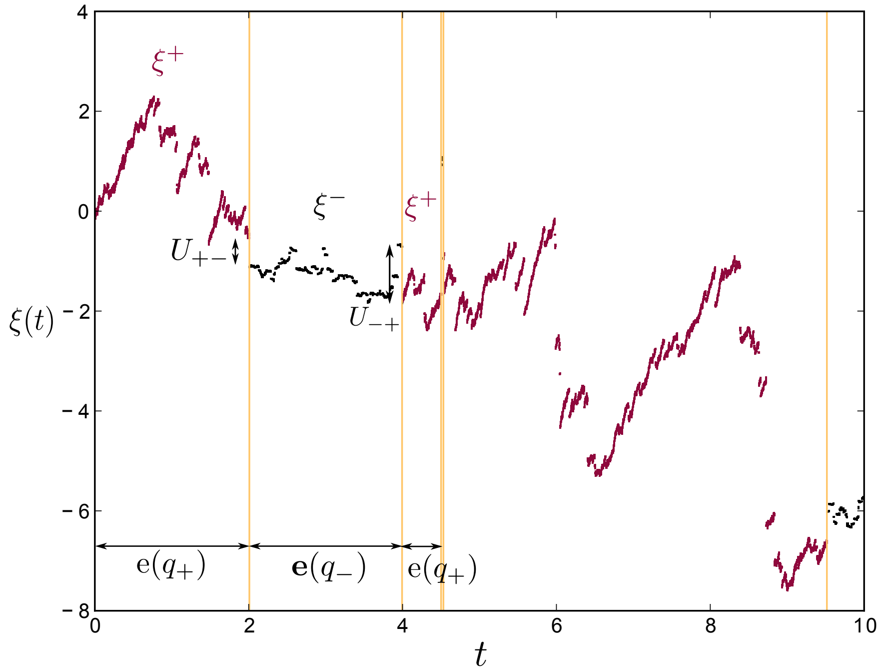
- $\mathbb{R}$-valued Markov process
- with initial measures P_x, $x \neq 0$
- 0 an absorbing state
- satisfying the scaling property

$$(cX_{tc^{-\alpha}})_{t\geq 0}|_{\mathrm{P}_x} \stackrel{d}{=} X|_{\mathrm{P}_{cx}}, \qquad x \neq 0,\ c > 0$$

Assume X jumps over zero in both directions

Two state $(+,-)$ Markov additive process

Lévy processes ξ^+,ξ^-; clock rates q_+,q_-; jumps U_{+-},U_{-+}
Markov additive process $(\xi(t), J(t))_{t\geq 0}$



MAP: characterisation

There exists a matrix $F(z) = \begin{pmatrix} F_{++}(z) & F_{+-}(z) \\ F_{-+}(z) & F_{--}(z) \end{pmatrix}$ such that

$$\left(e^{tF(z)}\right)_{ij} = \mathbb{E}\left[e^{z\xi(t)}; J(t) = j \mid J(0) = i,\, \xi(0) = 0\right].$$

F is determined by the components:

$$F(z) = \begin{pmatrix} \psi_+(z) & 0 \\ 0 & \psi_-(z) \end{pmatrix} + \begin{pmatrix} -q_+ & q_+ G_{+-}(z) \\ q_- G_{-+}(z) & -q_- \end{pmatrix}$$

where $e^{\psi_i(z)} = \mathbb{E}[e^{z\xi^i(1)}]$ and $G_{ij}(z) = \mathbb{E}[e^{zU_{ij}}]$.

Lamperti–Kiu transform (Chaumont, Pantí, Rivero)

α-rssMp X under $\mathrm{P}_{\pm 1}$	Two-state MAP (ξ, J)
$X_t = J(S(t)) \cdot \exp(\xi(S(t))$	$\xi(s) = \log(\lvert X_{T(s)}\rvert)$ $J(s) = \mathrm{sgn}(X_{T(s)})$
$S(t)$ inverse to $\int_0^{\cdot} e^{\alpha\xi(u)}\,\mathrm{d}u$	$T(s)$ inverse to $\int_0^{\cdot} \lvert X_u\rvert^{-\alpha}\,\mathrm{d}u$
X never hits zero X hits zero continuously $\Big\}$	$\Big\{$ $\xi \to \infty$ or ξ oscillates $\xi \to -\infty$
$T_0 = T(\infty) = \int_0^{\infty} e^{\alpha\xi(u)}\,\mathrm{d}u$	

Lamperti–Kiu transform: example

X: stable process killed on hitting zero (an α-rssMp)

Compute explicitly

$$F(z) = \frac{\Gamma(\alpha - z)\Gamma(1+z)}{\pi} \begin{pmatrix} -\sin(\pi(\alpha\hat{\rho} - z)) & \sin(\pi\alpha\hat{\rho}) \\ \sin(\pi\alpha\rho) & -\sin(\pi(\alpha\rho - z)) \end{pmatrix}$$

(hypergeometric Lévy processes and 'log-Pareto' jump distributions)

Exponential functional

Recall:

$$T_0 = I(\alpha\xi) := \int_0^\infty e^{\alpha\xi(u)}\,\mathrm{d}u.$$

How to characterise the exponential functional of ξ?
Let

$$\mathcal{M}(s) = \begin{pmatrix} \mathcal{M}_+(s) \\ \mathcal{M}_-(s) \end{pmatrix} = \begin{pmatrix} \mathbb{E}[I(\alpha\xi)^{s-1} \mid J(0) = +] \\ \mathbb{E}[I(\alpha\xi)^{s-1} \mid J(0) = -] \end{pmatrix},$$

the Mellin transform of $I(\alpha\xi)$.

Cramér condition & exponential functional

Whenever $F(z)$ is defined, it has a simple real eigenvalue $k(z)$ with maximal real part. We require:

Assumption (Cramér condition)

There exists some $\theta > 0$, such that F is defined on $(0, \theta + \epsilon)$ and $k(\theta) = 0$.

Proposition

Provided (ξ, J) satisfies the θ-Cramér condition,

$$\mathcal{M}(s+1) = -s(F(\alpha s))^{-1}\mathcal{M}(s),$$

for $\operatorname{Re} s \in (0, 1 + \theta/\alpha)$.

Solution: Mellin transform

Theorem

For $\operatorname{Re} s \in (-1/\alpha, 2 - 1/\alpha)$,

$$E_1[T_0^{s-1}] = \frac{\sin\left(\frac{\pi}{\alpha}\right)}{\sin(\pi\hat{\rho})} \frac{\sin\left(\pi\hat{\rho}(1-\alpha+\alpha s)\right)}{\sin\left(\frac{\pi}{\alpha}(1-\alpha+\alpha s)\right)} \frac{\Gamma(1+\alpha-\alpha s)}{\Gamma(2-s)}.$$

Alternative derivation, cf. also Letemplier & Simon (2013)

It is known that

$$E_1\left[e^{-\lambda T_0}\right] = \frac{u^\lambda(-1)}{u^\lambda(0)},$$

where

$$u^\lambda(x)\,dx = \int_0^\infty e^{-\lambda t} P(X_t \in dx)\,dt.$$

Trick:

$$E_1\left[T_0^{s-1}\right] = \frac{1}{\Gamma(1-s)} \int_0^\infty \frac{u^\lambda(-1)}{u^\lambda(0)} \lambda^{-s}\,d\lambda$$

and

$$u^\lambda(x) = -\frac{1}{\pi} \operatorname{Im}\left[\int_0^\infty \frac{e^{tx}}{\lambda + t^\alpha e^{i\pi\alpha\hat{\rho}}}\,dt\right]$$

Solution: density

Let $\mathrm{P}_1(T_0 \in \mathrm{d}t) = p(t)\,\mathrm{d}t$.

Theorem

For a dense, full-measure set of $\alpha \in (1,2)$:

$$p(t) = \frac{\sin\left(\frac{\pi}{\alpha}\right)}{\pi \sin(\pi\hat{\rho})} \sum_{k\geq 1} \frac{\sin(\pi\hat{\rho}(k+1))\sin(\frac{\pi}{\alpha}k)}{\sin(\frac{\pi}{\alpha}(k+1))} \frac{\Gamma(\frac{k}{\alpha}+1)}{k!}(-1)^{k-1} t^{-1-\frac{k}{\alpha}}$$
$$- \frac{\sin\left(\frac{\pi}{\alpha}\right)^2}{\pi \sin(\pi\hat{\rho})} \sum_{k\geq 1} \frac{\sin(\pi\alpha\hat{\rho}k)}{\sin(\pi\alpha k)} \frac{\Gamma\left(k-\frac{1}{\alpha}\right)}{\Gamma(\alpha k - 1)} t^{-k-1+\frac{1}{\alpha}}.$$

Application: conditioning to avoid zero. Cf. Pantí (2012)

Define

$$h(x) = \begin{cases} -\Gamma(1-\alpha)\dfrac{\sin(\pi\alpha\hat{\rho})}{\pi}|x|^{\alpha-1}, & x > 0, \\ -\Gamma(1-\alpha)\dfrac{\sin(\pi\alpha\rho)}{\pi}|x|^{\alpha-1}, & x < 0, \end{cases}$$

h is invariant for the stable process X killed on hitting zero. Let

$$\mathrm{P}^{\updownarrow}_x(\Lambda) = \frac{1}{h(x)}\mathrm{E}[h(X_t)\mathbb{1}_\Lambda;\ t < T_0], \qquad \Lambda \in \mathcal{F}_t,\ x \neq 0.$$

This is the process conditioned to avoid zero: for any stopping time T and $\Lambda \in \mathcal{F}_T$, and $x \neq 0$,

$$\lim_{q\downarrow 0} \mathrm{P}_x(\Lambda, T < \mathbf{e}_q \mid T_0 > \mathbf{e}_q) = \mathrm{P}^{\updownarrow}_x(\Lambda), \qquad \mathbf{e}_q \sim \mathrm{Exp}(q).$$

Application: conditioning to avoid zero

Write n for the excursion measure of X away from zero, and ζ for the lifetime of the excursion.

Proposition

For $x \neq 0$,

$$h(x) = \lim_{s \to \infty} \frac{\mathrm{P}_x(T_0 > s)}{n(\zeta > s)},$$

and for any stopping time T such that $\mathbb{E}_x[T] < \infty$, and $\Lambda \in \mathcal{F}_T$,

$$\mathrm{P}_x^{\updownarrow}(\Lambda) = \lim_{s \to \infty} \mathrm{P}_x(\Lambda | T_0 > T + s).$$

Further reading

A. Kuznetsov, A. E. Kyprianou, J. C. Pardo, A. R. Watson
The hitting time of zero for a stable process
arXiv:1212.5153 [math.PR]. To appear, *EJP*.

Thank you!

The 'dense subset' ...

Let $\|x\| = \min_{n\in\mathbb{Z}}|x - n|$, and

$$\mathcal{L} = \mathbb{R} \setminus (\mathbb{Q} \cup \{x \in \mathbb{R} : \lim_{n\to\infty} \frac{1}{n} \ln\|nx\| = 0\}).$$

$\mathcal{L}$ is small – it has Hausdorff dimension zero.

The previous result holds for $\alpha \notin \mathbb{Q} \cup \mathcal{L}$.

. . . and what happens outside

Define

$$\mathcal{K}(\alpha) = \{N \in \mathbb{N} : \|(N - \tfrac{1}{2})\alpha\| > \exp(-\tfrac{\alpha-1}{2}(N-2)\ln(N-2))\}$$

For $\alpha \notin \mathbb{Q}$, it holds that

$$p(t) = \lim_{\substack{N\in\mathcal{K}(\alpha)\\ N\to\infty}} \Bigg[\frac{\sin(\frac{\pi}{\alpha})}{\pi\sin(\pi\hat{\rho})} \sum_{1\le k<\alpha(N-\frac{1}{2})-1} \frac{\sin(\pi\hat{\rho}(k+1))\sin(\frac{\pi}{\alpha}k)}{\sin(\frac{\pi}{\alpha}(k+1))} \times \frac{\Gamma(\frac{k}{\alpha}+1)}{k!}(-1)^{k-1}t^{-1-\frac{k}{\alpha}} - \frac{\sin^2(\frac{\pi}{\alpha})}{\pi\sin(\pi\hat{\rho})} \sum_{1\le k<N} \frac{\sin(\pi\alpha\hat{\rho}k)}{\sin(\pi\alpha k)} \frac{\Gamma(k-\frac{1}{\alpha})}{\Gamma(\alpha k-1)} t^{-k-1+\frac{1}{\alpha}} \Bigg].$$