

Growth-fragmentation models, random and deterministic

Alex Watson¹

Joint work with Jean Bertoin

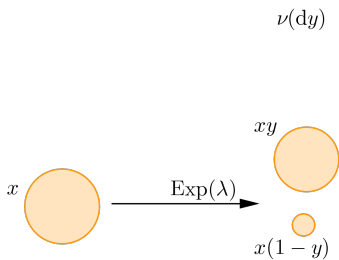
¹University of Manchester

1 February 2016
Oxford

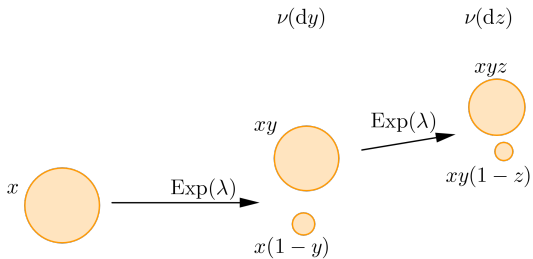
A finite-activity model



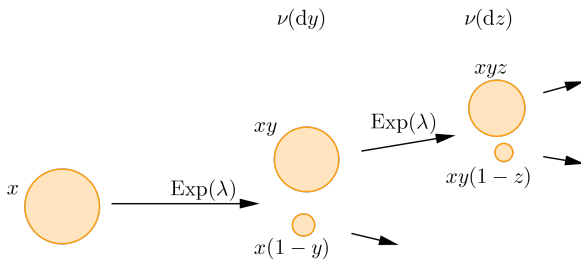
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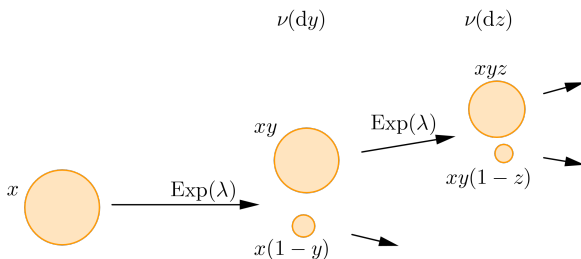
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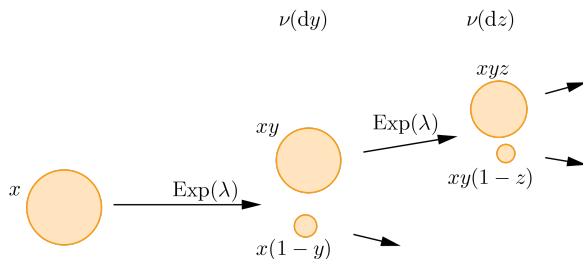


A finite-activity model



■ $\langle \mu_t, f \rangle = \int f(x) \mu_t(dx) = \mathbb{E} \left[\sum_{\substack{u \text{ particle} \\ \text{alive at } t}} f(\text{size}(u)) \right]$

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- $K(dy) = \lambda \nu(dy)$

The fragmentation equation

$$\partial_t \langle \mu_t, f \rangle = \left\langle \mu_t, \int_{[\frac{1}{2}, 1)} \{f(xy) + f(x(1-y)) - f(x)\} K(dy) \right\rangle,$$
$$f \in C_c^\infty(0, \infty),$$
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- Probabilistic approach: Haas, Banasiak
- Analytic/applied approaches: Doumic, Escobedo, Gabriel, ...

Questions

- Existence and representation
- Uniqueness
- Asymptotics
- Non-existence

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Fragmentation processes ($K < \infty$)

Suppose that $K[1/2, 1) = \lambda < \infty$.

- Start with a single 'fragment' of size x
- After an $\text{Exp}(\lambda)$ clock rings, it dies and produces offspring
- With probability $K(dy)/\lambda$, create new fragments, of size xy and $x(1 - y)$
- They evolve independently

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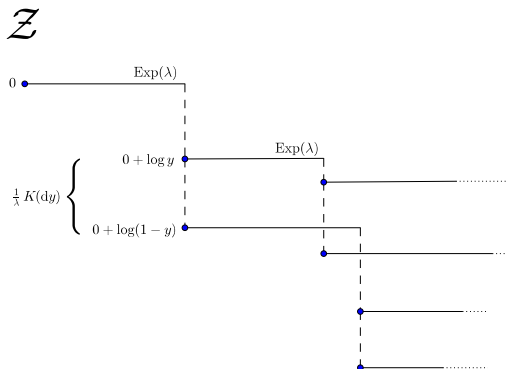
See this as a point process:

$$\mathcal{Y}(t) = \sum_{u \text{ fragments}} \delta_{\text{size}(u)} \mathbb{1}_{\{u \text{ alive at time } t\}}.$$

Point process perspective

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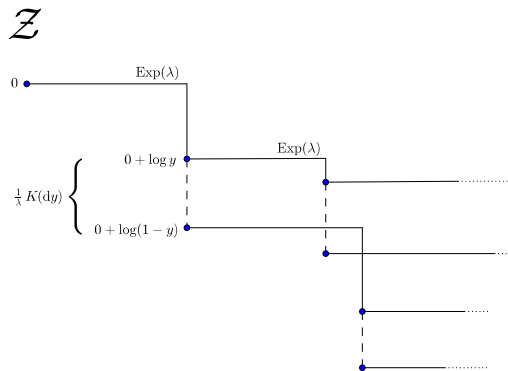


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is a compound Poisson process with immigration

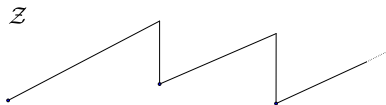


Compensated fragmentation processes, $\int (1-y)^2 K(dy) < \infty$

- Generalise \mathcal{Z}
- Create a Lévy process whose Lévy measure is the image of $K \circ \log^{-1}$
- It will have no positive jumps but it will not be decreasing
- Again, at every jump of size $\log y$, immigrate a new particle at relative position $\log(1-y)$
- Define $\mathcal{Y} = \mathcal{Z} \circ \exp^{-1}$

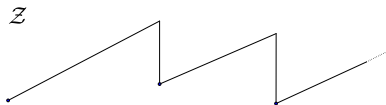
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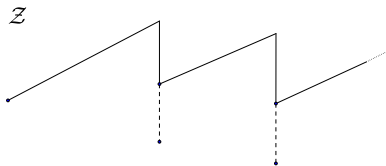
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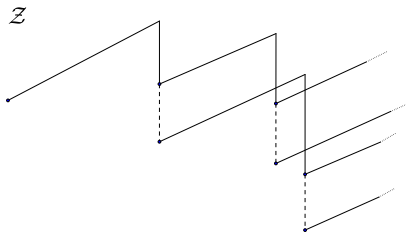
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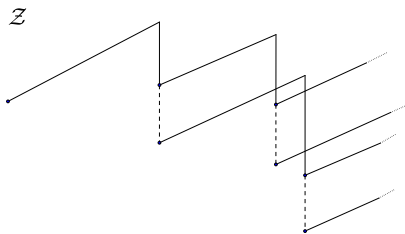
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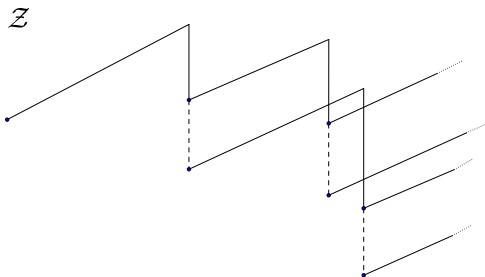


If $\int (1-y) K(dy) < \infty$, it is an 'exchangeable fragmentation' with growth/erosion.

Spines and solutions

We pick out a single trajectory from the fragmentation process \mathcal{Y} .

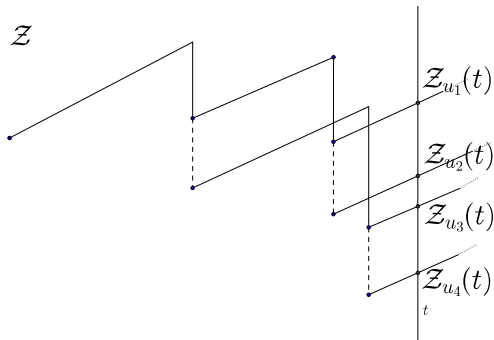
- Cut the process at time t and examine the particles
- Pick a particle u with probability $\propto \mathcal{Y}_u(t)^\omega = \exp\{\omega \mathcal{Z}_u(t)\}$ (with any $\omega \geq 2$)
- Trace its trajectory back
- It forms an exponential Lévy process; call it ξ



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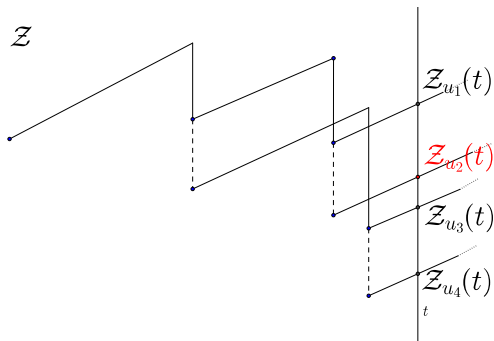
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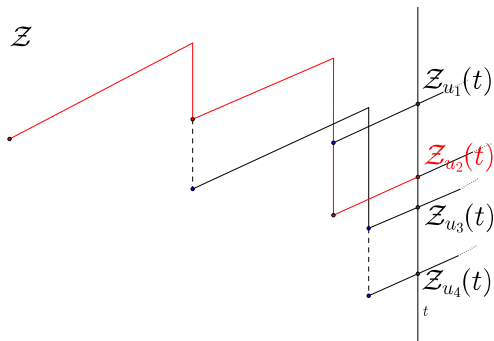
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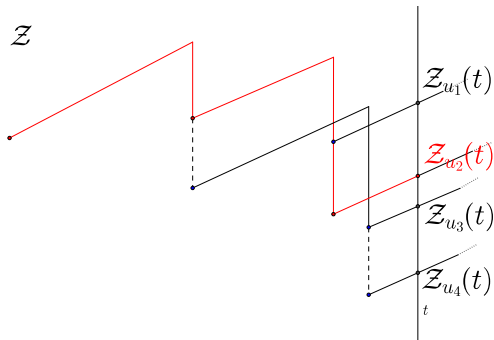
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Theorem

Let

$$\langle \mu_t, f \rangle = \mathbb{E}_{\delta_1} \left[\sum_u f(\mathcal{Y}_u(t)) \right] = e^{ct} \mathbb{E}_1[\xi(t)^{-\omega} f(\xi(t))].$$

This is the unique solution of the above equation with $\mu_0 = \delta_1$ and domain $f \in C_c^\infty(0, \infty)$.

Self-similarity

A generalisation:

$$\partial_t \langle \mu_t, f \rangle = \left\langle \mu_t, x^\alpha \left[axf'(x) + \int_{[\frac{1}{2}, 1)} \{f(xy) + f(x(1-y)) - f(x) + (1-y)xf'(x)\} K(dy) \right] \right\rangle,$$

- $\alpha \in \mathbb{R}$
- Things get more interesting!

Self-similar spines

- The role of the spine is played by a positive, self-similar Markov process with index $-\alpha$
- Take the old spine ξ and apply the *Lamperti transform*:
- Let

$$T(s) = \int_0^s \xi(u)^{-\alpha} du,$$

and write S for its inverse

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Solutions, $\alpha < 0$

$$\kappa(q) = aq + \int_{[\frac{1}{2}, 1)} \{y^q + (1-y)^q - 1 + (1-y)q\} K(dy)$$

- Assume there exists $\omega \in \mathbb{R}$ with $\kappa(\omega) = 0$ and $\kappa'(\omega) > 0$.
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Theorem ($\alpha < 0$)

- *There exists a solution (μ_t) to the self-similar growth-fragmentation equation, such that $\langle \mu_t, x^\omega \rangle \equiv 1$ and $\mu_0 = \delta_1$. It is given by $\langle \mu_t, f \rangle = \mathbb{E}_1[X(t)^{-\omega} f(X_t)]$.*

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- *There exists another solution (γ_t) , such that $\langle \gamma_t, x^\omega \rangle \equiv 1$ for $t > 0$ but $\gamma_0 = 0$. It is given by $\langle \gamma_t, f \rangle = \mathbb{E}_0[X(t)^{-\omega} f(X_t)]$.*

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Corollary (to the proof)

*If we require the growth-fragmentation equation to hold for all functions x^q , $q \geq \omega + \alpha$, then (μ_t) is the **unique solution** with $\mu_0 = \delta_1$.*

Asymptotics, $\alpha < 0$

Proposition

For $f \in C_b(0, \infty)$,

$$\int f(t^{-1/|\alpha|}x)x^\omega \mu_t(dx) \rightarrow \int f(x)x^\omega \gamma_1(dx), \quad t \rightarrow \infty.$$

Asymptotics, $\alpha < 0$

Suppose there exist $\rho < \omega$ such that $\kappa(\rho) = \kappa(\omega) = 0$.

Proposition

For $f \in C_0(0, \infty)$,

$$\frac{\int f(t^{-1}x^{|\alpha|})x^\rho \mu_t(dx)}{g(t)} \rightarrow \int f(x)v(dx), \quad t \rightarrow \infty,$$

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where $g \in RV(-\sigma)$, $\sigma = (\omega - \rho)/|\alpha|$, and v is related to factorisations of the exponential functional; cf. Haas–Rivero.

Solutions, $\alpha > 0$

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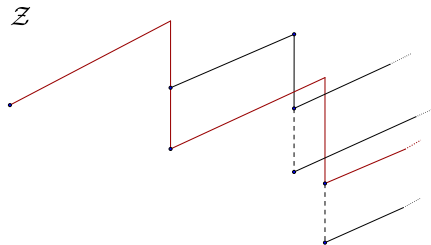
- *There exists a solution (μ_t) to the self-similar growth-fragmentation equation, such that $\langle \mu_t, x^\rho \rangle \equiv 1$ and $\mu_0 = \delta_1$. It is given by $\langle \mu_t, f \rangle = \mathbb{E}_1[X(t)^{-\rho} f(X_t)]$.*
- *Suppose $\kappa(\rho - \epsilon) < \infty$. Then there exists another solution (γ_t) , such that $\langle \gamma_t, x^\rho \rangle \equiv 1$ for $t > 0$ but $\gamma_0 = 0$. It is given by $\langle \gamma_t, f \rangle = \mathbb{E}_{+\infty}[X(t)^{-\rho} f(X_t)]$.*

'Explosion': self-similar fragmentations ($\alpha < 0$)

What goes wrong when there is no ω with $\kappa(\omega) = 0$?

'Explosion': self-similar fragmentations ($\alpha < 0$)

For a 'ray' v ,

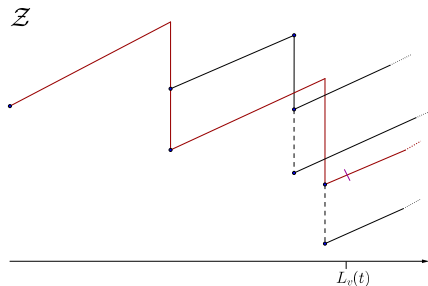


'Explosion': self-similar fragmentations ($\alpha < 0$)

For a 'ray' v , we define a functional

$$\lambda_v(t) = \int_0^t e^{-\alpha Z_v(s)} ds,$$

and write L_v for its inverse. This is a 'stopping line' time-change.

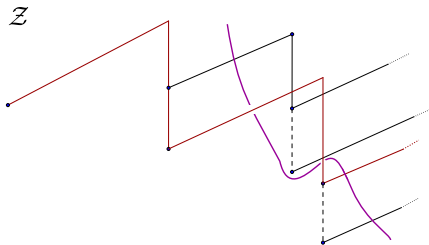


'Explosion': self-similar fragmentations ($\alpha < 0$)

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The self-similar fragmentation process is then

$$\mathcal{Y}^{(\alpha)}(t) = \mathcal{Y}(L(t)) = \mathcal{Z}(L(t)) \circ \exp^{-1}.$$

Explosion, $\alpha < 0$

Even in the simplest case (finite fragmentation), we have:

Proposition

If $\kappa = 0$ has no solutions then, for any $b > 0$, there exists a random time S such that $\#\{u : \mathcal{Y}_u^{(\alpha)}(S) \in [1, 1 + b]\} = \infty$*

Open questions

- Biased mass functions ($\alpha \neq 0$)
- Strengthen non-existence result
- Minimal solutions
- Process variant of 'starting from zero'
- Many other questions about compensated fragmentations

Further reading



J. Bertoin

Compensated fragmentation processes and limits of dilated fragmentations

[hal-00966190v2](#)



J. Bertoin, A. R. Watson

Probabilistic aspects of critical growth-fragmentation equations

[arXiv:1506.09187 \[math.PR\]](#)

Thank you!