Growth-fragmentation models, random and deterministic

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Joint work with Jean Bertoin

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The fragmentation equation

$$\partial_t \langle \mu_t, f \rangle = \left\langle \mu_t, \int_{[\frac{1}{2},1)} \{ f(xy) + f(x(1-y)) - f(x) \} \, K(\mathrm{d}y) \right\rangle, \quad f \in C_c^\infty(0,\infty),$$

$$\mu_0 = \delta_1$$

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- Probabilistic approach: Haas, Banasiak
- Analytic/applied approaches: Doumic, Escobedo, Gabriel, ...

Questions

- Existence and representation
- Uniqueness
- Asymptotics
- Non-existence

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Fragmentation processes ($K < \infty$)

Suppose that $K[1/2, 1) = \lambda < \infty$.

- Start with a single ‘fragment’ of size x
- After an $\mathrm{Exp}(\lambda)$ clock rings, it dies and produces offspring
- With probability $K(\mathrm{d}y)/\lambda$, create new fragments, of size xy and $x(1-y)$
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See this as a point process:

$$\mathcal{Y}(t) = \sum_{u \text{ fragments}} \delta_{\mathrm{size}(u)} \mathbb{1}_{\{u \text{ alive at time } t\}}.$$

Point process perspective

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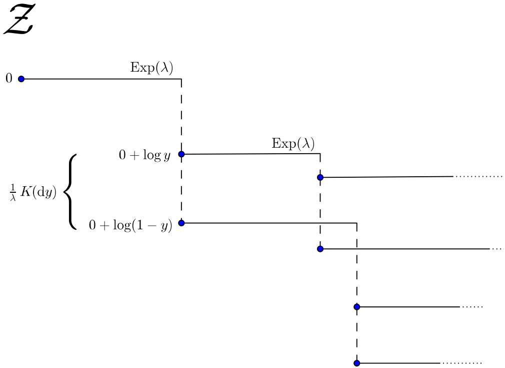

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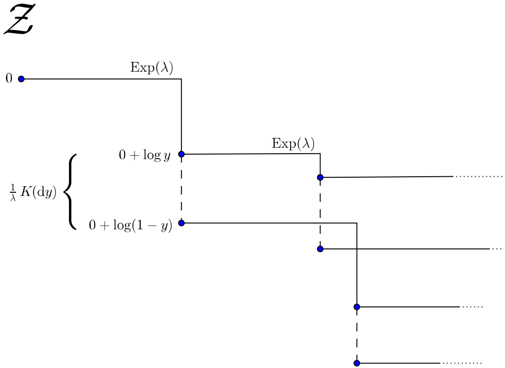
is a compound Poisson process with immigration



Compensated fragmentation processes, $\int (1-y)^2 \, K(\mathrm{d}y) < \infty$

- Generalise $\mathcal{Z}$
- Create a Lévy process whose Lévy measure is the image of $K \circ \log^{-1}$
- It will have no positive jumps but it will not be decreasing
- Again, at every jump of size $\log y$, immigrate a new particle at relative position $\log(1-y)$
- Define $\mathcal{Y} = \mathcal{Z} \circ \exp^{-1}$

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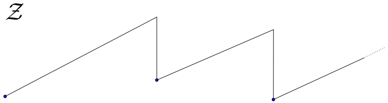

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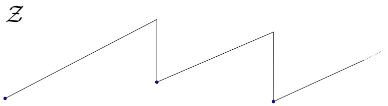

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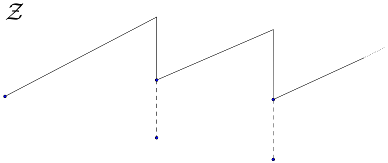

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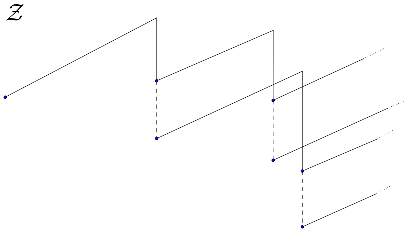

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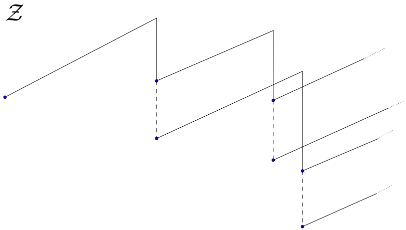
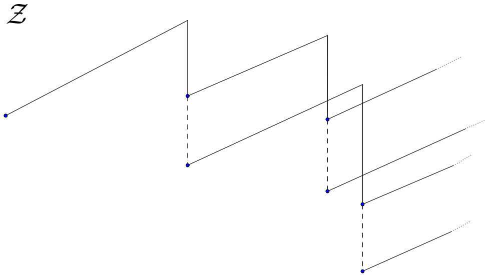


If $\int (1-y)\, K(\mathrm{d}y) < \infty$, it is an 'exchangeable fragmentation' with growth/erosion.

Spines and solutions

We pick out a single trajectory from the fragmentation process $\mathcal{Y}$.

- Cut the process at time t and examine the particles
- Pick a particle u with probability $\propto \mathcal{Y}_u(t)^\omega = \exp\{\omega \mathcal{Z}_u(t)\}$ (with any $\omega \geq 2$)
- Trace its trajectory back
- It forms an exponential Lévy process; call it ξ



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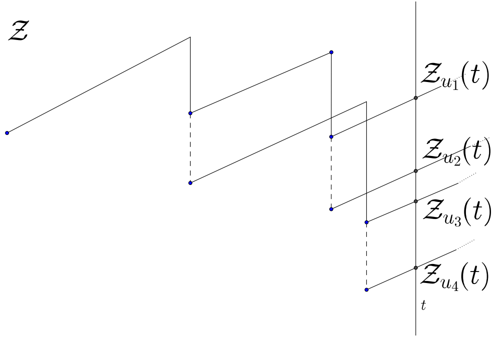

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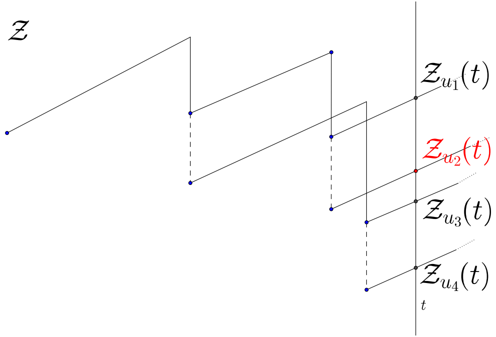

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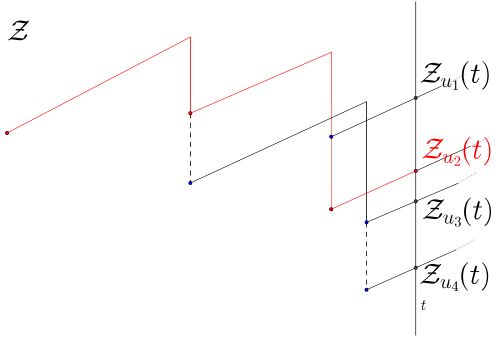

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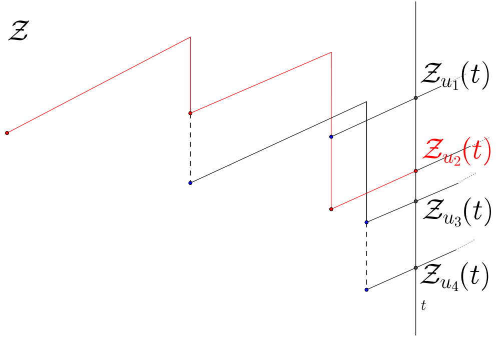

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Theorem

Let

$$\langle \mu_t, f\rangle = \mathbb{E}_{\delta_1}\Big[\sum_u f(\mathcal{Y}_u(t))\Big] = e^{ct}\mathbb{E}_1[\xi(t)^{-\omega} f(\xi(t))].$$

This is the unique solution of the above equation with $\mu_0 = \delta_1$ and domain $f \in C_c^\infty(0,\infty)$.

Self-similarity

A generalisation:

$$\partial_t \langle \mu_t, f \rangle = \left\langle \mu_t, x^\alpha \left[axf'(x) + \int_{[\frac{1}{2},1)} \{ f(xy) + f(x(1-y)) - f(x) + (1-y)xf'(x) \} \, K(\mathrm{d}y) \right] \right\rangle,$$

- $\alpha \in \mathbb{R}$
- Things get more interesting!

Self-similar spines

- The role of the spine is played by a positive, self-similar Markov process with index $-\alpha$
- Take the old spine ξ and apply the *Lamperti transform*:
- Let

$$T(s) = \int_0^s \xi(u)^{-\alpha} \, \mathrm{d}u,$$

  and write S for its inverse
- The self-similar spine is the process $X(t) = \xi(S(t))$

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Solutions, $\alpha < 0$

$$\kappa(q) = aq + \int_{[\frac{1}{2},1)} \{y^q + (1-y)^q - 1 + (1-y)q\} \, K(\mathrm{d}y)$$

- Assume there exists $\omega \in \mathbb{R}$ with $\kappa(\omega) = 0$ and $\kappa'(\omega) > 0$.
- X is the self-similar spine with weighting ω

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Theorem ($\alpha < 0$)

- *There exists a solution (μ_t) to the self-similar growth-fragmentation equation, such that $\langle \mu_t, x^\omega \rangle \equiv 1$ and $\mu_0 = \delta_1$. It is given by $\langle \mu_t, f \rangle = \mathbb{E}_1[X(t)^{-\omega} f(X_t)]$.*

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Corollary (to the proof)

If we require the growth-fragmentation equation to hold for all functions x^q, $q \geq \omega + \alpha$, then (μ_t) is the unique solution with $\mu_0 = \delta_1$.

Asymptotics, $\alpha < 0$

Proposition

For $f \in C_b(0, \infty)$,

$$\int f(t^{-1/|\alpha|} x) x^\omega \mu_t(dx) \to \int f(x) x^\omega \gamma_1(dx), \qquad t \to \infty.$$

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Suppose there exist $\rho < \omega$ such that $\kappa(\rho) = \kappa(\omega) = 0$.

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For $f \in C_0(0, \infty)$,

$$\frac{\int f(t^{-1}x^{|\alpha|})x^{\rho}\mu_t(\mathrm{d}x)}{g(t)} \to \int f(x)\upsilon(\mathrm{d}x), \qquad t \to \infty,$$

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where $g \in RV(-\sigma)$, $\sigma = (\omega - \rho)/|\alpha|$, and υ is related to factorisations of the exponential functional; cf. Haas–Rivero.

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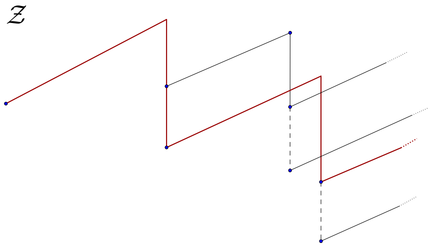
'Explosion': self-similar fragmentations ($\alpha < 0$)

What goes wrong when there is no ω with $\kappa(\omega) = 0$?

‘Explosion’: self-similar fragmentations ($\alpha < 0$)

For a ‘ray’ v,

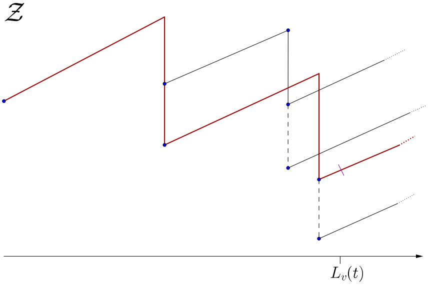


‘Explosion’: self-similar fragmentations ($\alpha < 0$)

For a ‘ray’ v, we define a functional

$$\lambda_v(t) = \int_0^t e^{-\alpha \mathcal{Z}_v(s)}\, ds,$$

and write L_v for its inverse. This is a ‘stopping line’ time-change.

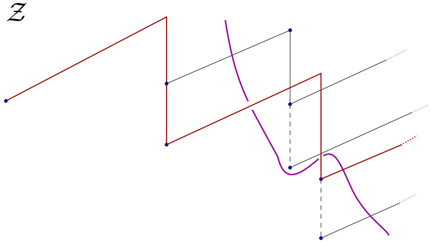


‘Explosion’: self-similar fragmentations ($\alpha < 0$)

For a ‘ray’ v, we define a functional

$$\lambda_v(t) = \int_0^t e^{-\alpha \mathcal{Z}_v(s)} \, \mathrm{d}s,$$

and write L_v for its inverse. This is a ‘stopping line’ time-change.



The self-similar fragmentation process is then

$$\mathcal{Y}^{(\alpha)}(t) = \mathcal{Y}(L(t)) = \mathcal{Z}(L(t)) \circ \exp^{-1}.$$

Explosion, $\alpha < 0$

Even in the simplest case (finite fragmentation), we have:

Proposition

If $\kappa = 0$ has no solutions then, for any $b > 0$, there exists a random time S such that $\#\{u : \mathcal{Y}_u^{(\alpha)}(S) \in [1, 1+b]\} = \infty$*

Open questions

- Biased mass functions ($\alpha \neq 0$)
- Strengthen non-existence result
- Minimal solutions
- Process variant of ‘starting from zero’
- Many other questions about compensated fragmentations

Further reading

J. Bertoin
Compensated fragmentation processes and limits of dilated fragmentations
hal-00966190v2

J. Bertoin, A. R. Watson
Probabilistic aspects of critical growth-fragmentation equations
arXiv:1506.09187 [math.PR]

Thank you!