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Spa2015@oxford-man.ox.ac.uk

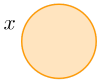
Growth-fragmentation models, random and deterministic

Alex Watson[1]

Joint work with Jean Bertoin

[1]University of Zurich

A finite-activity model



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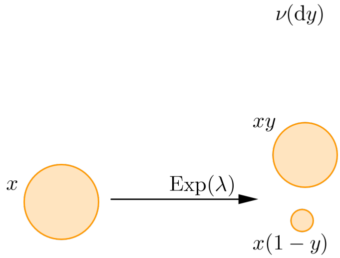

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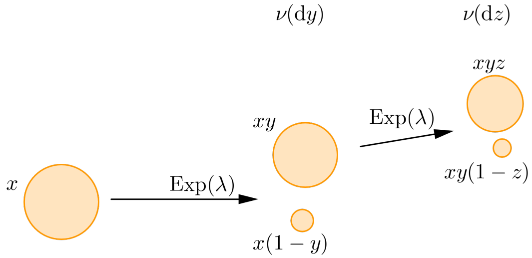

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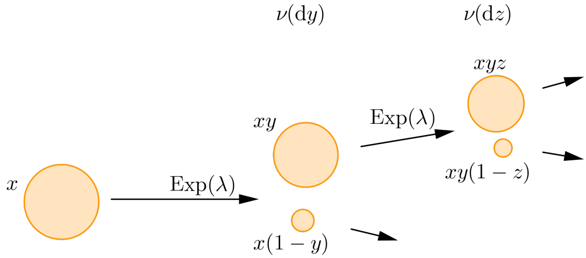

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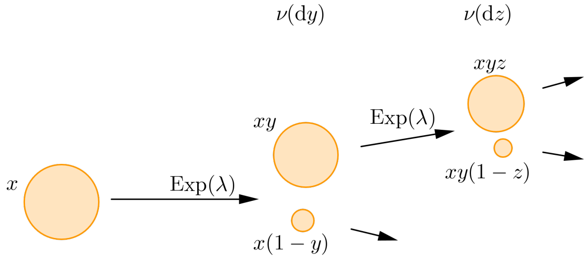

- $\langle \mu_t, f \rangle = \int f(x)\, \mu_t(\mathrm{d}x) = \mathbb{E}\Big[\sum_{u \text{ particle alive at } t} f(\mathrm{size}(u))\Big]$

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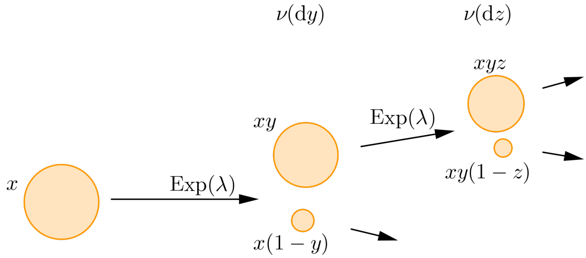

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- $K(\mathrm{d}y) = \lambda \nu(\mathrm{d}y)$

The pure fragmentation equation

$$\partial_t \langle \mu_t, f \rangle = \left\langle \mu_t, \int_{[\frac{1}{2},1)} \{ f(xy) + f(x(1-y)) - f(x) \} \, K(\mathrm{d}y) \right\rangle, \qquad f \in C_c^\infty(0,\infty),$$

$$\mu_0 = \delta_1$$

The growth-fragmentation equation

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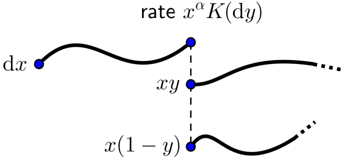
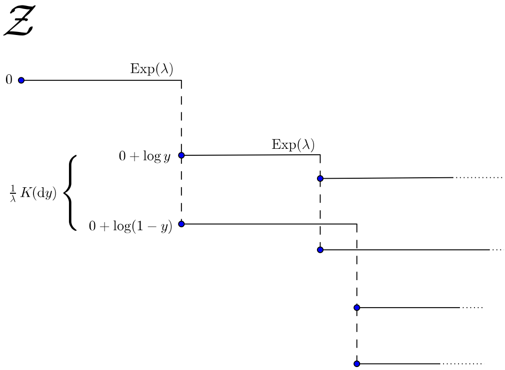


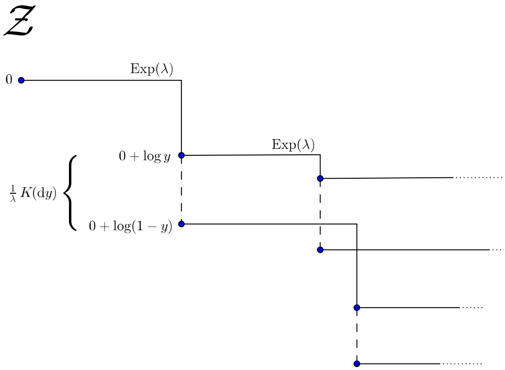
Fragmentation processes, $\alpha = 0$

The finite-activity model, on log-scale...



Fragmentation processes, $\alpha = 0$

The finite-activity model, on log-scale...
...is a compound Poisson process with immigration



Compensated fragmentation processes, $\int (1-y)^2 \, K(\mathrm{d}y) < \infty$

- We can build the general (log-scale) model.
- Pick a Lévy process with negative jumps
- At each jump of size $\log y$, immigrate a new particle at relative position $\log(1-y)$

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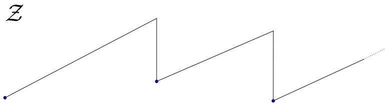

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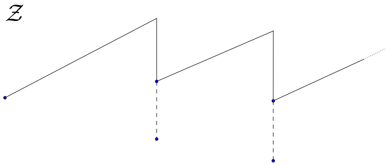

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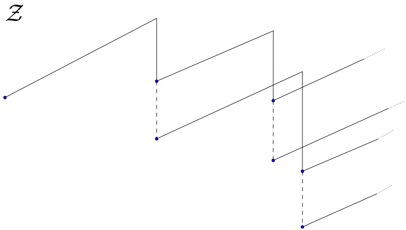

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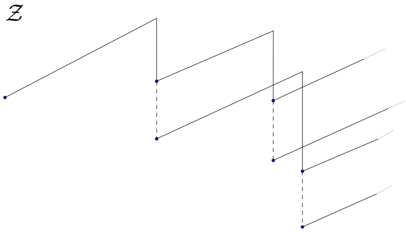

If $\int (1-y)\, K(\mathrm{d}y) < \infty$, it is an 'exchangeable fragmentation' with growth/erosion.

Solutions

Theorem ($\alpha = 0$)

The measure

$$\langle \mu_t, f \rangle = \mathbb{E}_{\delta_1}\Bigg[\sum_{\substack{u \text{ particle} \\ \text{alive at } t}} f(\text{size}(u)) \Bigg]$$

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- *An analogous solution (μ_t) with $\mu_0 = \delta_1$ exists.*

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Theorem ($\alpha \neq 0$)

Under a Malthusian condition,

- *An analogous solution (μ_t) with $\mu_0 = \delta_1$ exists.*
- *There is another solution (γ_t), with $\gamma_0 = 0$ but $\gamma_t \neq 0$ for all $t > 0$.*

Other work and open questions

- Asymptotics
- Non-existence (absent Malthusian condition)
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- Asymptotics
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- Minimal solutions
- Biased mass functions
- Process variant of ‘starting from zero’
- Many other questions about compensated fragmentations

Further reading

J. Bertoin
Compensated fragmentation processes and limits of dilated fragmentations
hal-00966190v2

J. Bertoin, A. R. Watson
Probabilistic aspects of critical growth-fragmentation equations
arXiv:1506.09187 [math.PR]

Thank you!