Growth-fragmentation models, random and deterministic

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38th Conference on Stochastic Processes and their Applications, Oxford, 15 July 2015

Abstract. We look at models of fragmentation with growth. In such a model, one has a number of independent cells, each of which grows continuously in time until a fragmentation event occurs, at which point the cell splits into two child cells of a smaller mass. Each of the children is independent and behaves in the same way as its parent. The rate of fragmentation may be infinite, and fragmentation may be homogeneous (where the rate does not depend on the mass of the cell) or self-similar (where the rate is a power of the mass). This is a random model; looking at its mean-field behaviour gives the growth-fragmentation equation, which is a deterministic PDE. Variants of the equation have been studied recently from an analytic perspective, but we use a probabilistic approach, akin to that taken by Haas for the pure fragmentation equation. We describe probabilistic solutions to the equation, using the compensated fragmentation processes recently described by Bertoin. One interesting phenomenon is that, in some self-similar cases, we see spontaneous generation of positive solutions from zero initial mass.

Based on joint work with Jean Bertoin (University of Zurich); see [3].

About these notes. This document is generated from my `beamer` presentation based on notes left in the source. My hope is that the reader who wishes they had attended my talk, but had an important prior engagement with a grant proposal or a game of Minecraft, will be able to read this and obtain a similar experience without the inconvenience of sitting in a seminar room. It should read somewhere between an informal talk and a formal article, and as such there are likely to be some inaccuracies, partly because of the informality of the language, and partly because the notes were mostly written from memory.

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This document is probably best read along with the accompanying slides, but ‘flattened’ (and somewhat mangled) versions are included for convenience, demarcated by horizonal rules.

Let’s go.

We begin with a discussion of the finite-activity model, which has been around in the probability literature for a long time, and goes back at least as far a Kolmogorov in 1941. For a discussion of the literature in the finite activity case and the exchangeable fragmentation framework, we refer to [1].

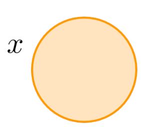
In this model, we have a collection of particles (cells in the body, fragments of ore in a mining tumbler, etc.) which are characterised only by their size (mass, diameter, etc.). No geometry is in play, so we are in a so-called ‘mean field’ situation.

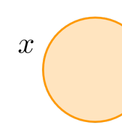
At time zero, we start with a single particle of size x. After some exponentially distribution time of rate λ, not depending on x, the particle breaks apart into two new particles of size xy and $x(1-y)$, where y is chosen according to some probability measure ν. These particles then evolve independently and with the same dynamics. Note that we have binary fragmentation and no loss of mass.

On the next slide, we will write down an equation for the mean measure of this model. This measure will be written μ_t, and is given as in the slide.

To unify notation (and allow the correct generalisation) we write K for $\lambda\nu$.

A finite-activity model





- $\langle \mu_t, f\rangle = \int f(x)\,\mu_t(\mathrm{d}x) = \mathbb{E}\Big[\sum_{\substack{u \text{ particle} \\ \text{alive at } t}} f(\mathrm{size}(u))\Big]$

- $K(\mathrm{d}y) = \lambda\nu(\mathrm{d}y)$

With this definition of μ_t, it is not difficult to see that the evolution is governed by this partial differential equation in weak form. On the right-hand side, x is a dummy variable, and the interpretation is that, at rate $K(\mathrm{d}y)$, a particle of size x is removed $(-f(x))$ and two new particles of size xy and $x(1-y)$ are added $(+f(xy)+f(x(1-y)))$.

Also important on this slide is the domain of the equation, namely $C_\mathrm{c}^\infty(0,\infty)$, the space of smooth functions with compact support. In fact, this domain is sufficient to ensure that there is a unique solution of the PDE with the initial condition $\mu_0 = \delta_1$. (We could substitute any δ_x for the initial condition, since the equation is self-similar.)

The pure fragmentation equation

$$\partial_t\langle\mu_t, f\rangle = \left\langle \mu_t, \int_{[\frac{1}{2},1)} \{f(xy) + f(x(1-y)) - f(x)\}\, K(\mathrm{d}y) \right\rangle, \qquad f \in C_\mathrm{c}^\infty(0,\infty),$$

$$\mu_0 = \delta_1$$

We will generalise the pure, finite-activity fragmentation equation in three ways simultaneously.

We first add a term $axf'(x)$; this is a deterministic exponential growth (if $a > 0$) or exponential decay (if $a < 0$) term.

We then include a term $(1-y)xf'(x)$ in the intergral; this is a kind of compensation, and has the effect of permitting us to expand the acceptable choices of the measure K, because the right-hand side is now well-defined so long as $\int(1-y)^2\, K(\mathrm{d}y) < \infty$, whereas without the compensation term we would have required the condition $\int(1-y)\, K(\mathrm{d}y) < \infty$, which is stronger. The weaker moment condition essentially allows extremely high rates of fragmentation into very asymmetric child particles (where y is very close to 1.)

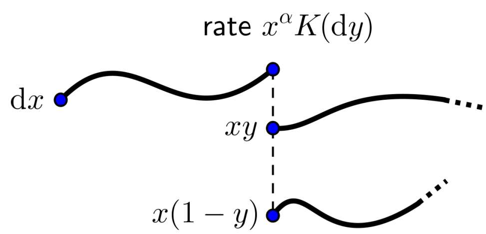
Finally, we include a factor x^α around everything, which makes the equation α-self-similar, in the sense that if $(\mu_t)_{t\geq 0}$ is a solution with initial condition $\mu_0 = \delta_1$, and if $\tilde{\mu}_t$ denotes the image of μ_t by the dilation $x \mapsto cx$, then $(\tilde{\mu}_{c^\alpha t})_{t\geq 0}$ is a solution with initial condition

$\tilde{\mu}_0 = \delta_c$. In terms of particles, it essentially means that a particle of size dx experiences a fragmentation with relative child size y at rate $x^\alpha K(\mathrm{d}y)$.

The growth-fragmentation equation

$$\partial_t \langle \mu_t, f \rangle = \left\langle \mu_t, x^\alpha \left[axf'(x) + \int_{[\frac{1}{2},1)} \{f(xy) + f(x(1-y)) - f(x) + (1-y)xf'(x)\} \, K(\mathrm{d}y) \right] \right\rangle,$$

- $a \in \mathbb{R}$
- Require only $\int (1-y)^2 \, K(\mathrm{d}y) < \infty$ (asymmetric children)
- $\alpha \in \mathbb{R}$



We plot on the first slide the sizes of particles in a finite-activity fragmentation process. Each lives an exponential time, before dying and producing children.

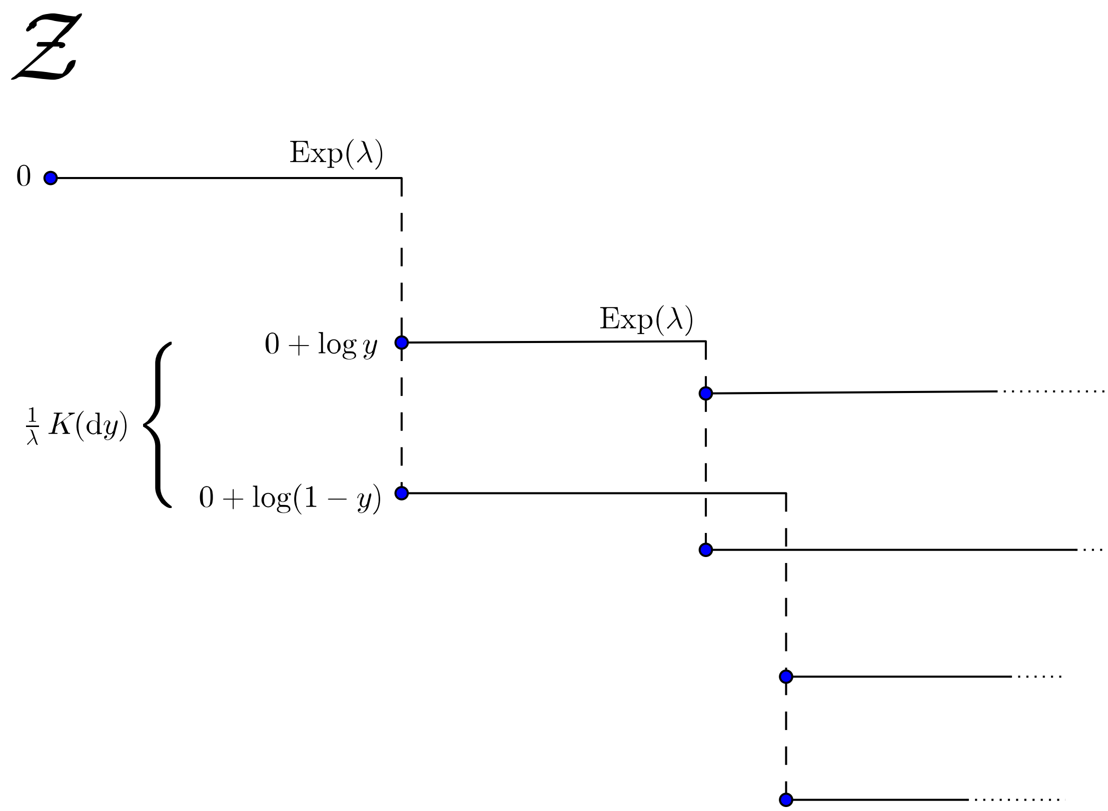
In the overlay, we systematically identify each particle with its largest child; this results (on log-scale) in a compound Poission process with jump rate λ and jump size K/λ, with immigration at the jumps.

This suggests an easy way to generalise the definition of a fragmentation process, if we think that a compound Poisson process is just the simplest example of a Lévy process.

Fragmentation processes, $\alpha = 0$

The finite-activity model, on log-scale...

...is a compound Poisson process with immigration

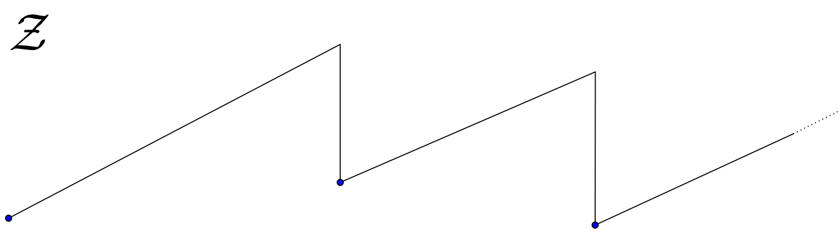


We make a new process on the log-scale by taking a spectrally negative Lévy process and immigrating at its jumps. This permits the moment condition $\int (1-y)^2 K(\mathrm{d}y) < \infty$, which is related to the condition $\int 1 \wedge x^2 \, \Pi(\mathrm{d}x) < \infty$ for a Lévy measure Π.

For the connoisseurs: if the stronger condition $\int (1-y) \, K(\mathrm{d}y) < \infty$ holds, which corresponds to the condition $\int 1 \wedge x \, \Pi(\mathrm{d}x) < \infty$ for a Lévy measure, we are left with a traditional exchangeable fragmentation with deterministic growth or erosion.

Compensated fragmentation processes, $\int (1-y)^2 \, K(\mathrm{d}y) < \infty$

- We can build the general (log-scale) model.
- Pick a Lévy process with negative jumps
- At each jump of size $\log y$, immigrate a new particle at relative position $\log(1-y)$



If $\int (1-y) \, K(\mathrm{d}y) < \infty$, it is an 'exchangeable fragmentation' with growth/erosion.

We now come to the results.

In the case $\alpha = 0$ (the homogeneous case), we have existence and uniqueness of solutions, and indeed a stochastic representation of μ as the mean of the compensated fragmentation process introduced on the previous slide.

In the case $\alpha \neq 0$, the situation is a little more complicated. First we introduce the function κ:

$$\kappa(q) \coloneqq aq^2 + (b-a)q + \int_{[1/2,1)} (y^q + (1-y)^q - 1 + q(1-y)) K(\mathrm{d}y), \qquad q \geq 0.$$

Under Malthusian conditions, essentially guaranteeing the existence of either one or two solutions to the equation $\kappa(q) = 0$, we have the existence of a solution (μ_t) with similar stochastic representation as in the homogeneous case (we use a positive, self-similar Markov process), but we also have another solution (γ_t) which starts from $\gamma_0 = 0$ but thereafter satisfies $\int x^\omega \gamma_t(\mathrm{d}x) = 1$ for all $t > 0$, where $\kappa(\omega) = 0$.

This of course means that uniqueness is impossible for $\alpha \neq 0$, because we can always add γ to any solution which we start with.

One remark which can be made about this case is that, when $\alpha < 0$, if we expand the domain of the growth-fragmentation equation to include all functions x^q, for $q \geq \omega + \alpha$, then we do recover uniqueness. For $\alpha > 0$ we have no such result.

Solutions

Theorem 1 ($\alpha = 0$)**.** *The measure*

$$\langle \mu_t, f \rangle = \mathbb{E}_{\delta_1}\Bigg[\sum_{\substack{u \text{ particle} \\ \text{alive at } t}} f(\mathrm{size}(u)) \Bigg]$$

is the unique solution with $\mu_0 = \delta_1$.
Theorem 2 ($\alpha \neq 0$)**.** *Under a Malthusian condition,*

- *An analogous solution (μ_t) with $\mu_0 = \delta_1$ exists.*
- *There is another solution (γ_t), with $\gamma_0 = 0$ but $\gamma_t \neq 0$ for all $t > 0$.*

We conclude with a short summary of the other work we completed on the equation. We looked at two asymptotic regimes for the solutions μ_t in the self-similar case; in the absence of a Malthusian condition, we showed that the stochastic model may experience blow-up (but did not attempt to transfer this is show that the deterministic model has no global solutions); and finally, the proof techniques essentially rely on finding spines of the fragmentation processes, though this is not entirely clear at present and we hope to present more work on this in the future.

To finish, some open questions are listed. The reader is also referred to the work of Bertoin et al. [4] and Shi [5].

Other work and open questions

- Asymptotics

- Non-existence (absent Malthusian condition)
- Spines of the fragmentation
- Minimal solutions
- Biased mass functions
- Process variant of 'starting from zero'
- Many other questions about compensated fragmentations

References

[1] J. Bertoin. *Random fragmentation and coagulation processes*, volume 102 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006. ISBN 978-0-521-86728-3; 0-521-86728-2. doi:10.1017/CBO9780511617768.

[2] J. Bertoin. Compensated fragmentation processes and limits of dilated fragmentations. Preprint, hal-00966190v2, 2014.

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[4] J. Bertoin, N. Curien, and I. Kortchemski. Random planar maps & growth-fragmentations. Preprint, arXiv:1507.02265v1 [math.PR], 2015.

[5] Q. Shi. Growth-fragmentation processes and the switching transformation of lévy processes. Preprint, 2015.